

A backward λ -lemma for the forward heatflow

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Dedicated to the memory of V.I. Arnol'd

Abstract

The inclination or λ -lemma is one of two fundamental tools in finite dimensional hyperbolic dynamics. In contrast to finite dimension, consider the forward semiflow on the loop space of a closed Riemannian manifold M provided by the heatflow. The main result is a backward λ -lemma for the heatflow near a hyperbolic fixed point x . There are the following novelties. Firstly, infinite versus finite dimension. Secondly, semiflow versus flow. Thirdly, new proof in the finite dimensional case. Fourthly and a priori most surprisingly, our λ -lemma moves the disk transversal to the unstable manifold backward in time although there is no backward flow. Finally, as a first application, we propose a simple method of calculating the Conley homotopy index of x .

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1 Introduction and main results

Assume M is a closed smooth manifold of dimension $n \geq 1$ equipped with a Riemannian metric g and the Levi-Civita connection ∇ . Throughout smooth means C^∞ smooth. We denote by ΛM the **loop space**, that is the Hilbert manifold of absolutely continuous loops $W^{1,2}(S^1, M)$. Here and throughout we identify $S^1 = \mathbb{R}/\mathbb{Z}$ and think of $x \in \Lambda M$ as a map $x : \mathbb{R} \rightarrow M$ which satisfies $x(t+1) = x(t)$. Pick a smooth function $V : S^1 \times M \rightarrow \mathbb{R}$ and set $V_t(q) := V(t, q)$.

It is well known that the Cauchy problem associated to the **heat equation**

$$\partial_s u - \nabla_t \partial_t u - \text{grad} V_t(u) = 0 \quad (1)$$

for maps $u : \mathbb{R} \rightarrow \Lambda M$, $s \mapsto u(s, \cdot) =: u_s$, admits a unique solution in forward time. The associated **semiflow** is a continuous map

$$\varphi : [0, \infty) \times \Lambda M \rightarrow \Lambda M$$

which is of class C^1 on $(0, \infty)$. Because (1) is the downward L^2 gradient equation of the action functional $\mathcal{S}_V(x) = \int_0^1 \left(\frac{1}{2} |\dot{x}(t)|^2 - V_t(x(t)) \right) dt$ on ΛM , the fixed points of the heatflow are the critical points of the action. The latter are (perturbed) closed geodesics, that is solutions $x : S^1 \rightarrow M$ to the ODE $-\nabla_t \dot{x} - \nabla V_t(x) = 0$. By $\text{ind}(x)$ we denote the Morse index of x . Nondegeneracy of the critical point x corresponds to hyperbolicity of the fixed point x .

Fix a nondegenerate critical point x of the action \mathcal{S}_V and set $c = \mathcal{S}_V(x)$. While the **stable manifold** $W^s(x)$ is defined in the usual way to be the set of all points which flow in forward time asymptotically into x , it is only known to be a manifold locally near x and of infinite dimension; see section 2.5. In contrast, without a backward flow the definition of the **unstable manifold** $W^u(x)$ becomes somewhat awkward: It is the set of endpoints of all heatflow trajectories parametrized by $(-\infty, 0]$ and emanating at time $-\infty$ from x . On the other hand, the definition lends itself to define a backward flow on $W^u(x)$. Furthermore, the unstable manifold is globally embedded and, most importantly, its dimension is given by $\text{ind}(x)$ and therefore finite; see e.g. [W10b]. *It is this finite dimensionality which is one of two pillars on which this paper is based.* The key consequence is smoothness of every $\gamma \in W^u(x)$; see remark 1.6.

1.1 History of the present paper

In finite dimensional hyperbolic dynamics there are two fundamental tools, the Grobman-Hartman theorem [G59, H60] and the λ -lemma [P69]. While the first is powerful concerning topological questions the latter reigns in the differentiable world. It even implies the former. The **λ -lemma** asserts that, roughly speaking, the backward flow applied to any disk D transversal to the unstable manifold and of complementary dimension converges in C^1 to the local stable manifold, see figure 1, and similarly for the forward flow. For a beautiful presentation we refer to and recommend [PM82]. Since convergence is in C^1 , the λ -lemma is also called the **inclination lemma**.

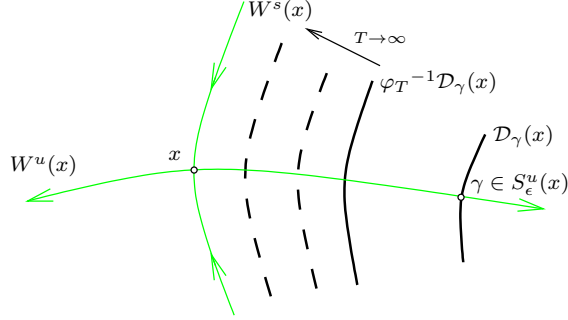


Figure 1: Fiber preimage converges in C^1 locally near x to stable manifold

The second pillar on which this paper is based is the replacement of the absent backward flow on the loop space by the family of preimages $s \mapsto \varphi_s^{-1}D$. This idea was born when we attempted to construct a Morse filtration of the loop space applying the method of Abbondandolo and Majer [AM06]. Their method builds on open sets being mapped to open sets under a forward flow. This is not true for φ_s – from the viewpoint of topology the heat semiflow is useless. The way out was the simple observation that by continuity of φ_s preimages of open sets are open. Unfortunately, still the Abbondandolo-Majer method would not apply, because things were moving in the wrong direction now. However, bringing in Conley theory we were able to construct a Morse filtration for semiflows [W12]. The key calculation is based on the backward λ -lemma and discussed in section 1.3. In other words, we came across the needs for a backward λ -lemma accidentally while solving another problem – a phenomenon addressed by Arnol'd over and over again in his marvelous books and papers.

To go from ΛM to $W^{k,p}(S^1, M)$ and to other semilinear parabolic PDEs is straightforward and will be carried out elsewhere; see remark 1.5 (i).

1.2 Main results

Assume $\mathcal{D}_\gamma(x)$ is a disk which intersects the unstable manifold of x transversally in a point γ near x . Our main goal is to prove that the preimage $\varphi_T^{-1}\mathcal{D}_\gamma(x)$ converges, as $T \rightarrow \infty$, uniformly in C^1 and locally near x to the stable manifold $W^s(x)$; see figure 1. In fact, we prove right away a family version where \mathcal{D} is fibered over a descending sphere $S_\varepsilon^u(x) = W^u(x) \cap \{\mathcal{S}_V = c - \varepsilon\}$.

Since the λ -lemma is a local result we choose a local parametrization

$$\Phi := \exp_x : X = T_x \Lambda M = W^{1,2}(S^1, x^* TM) \supset \mathcal{U} \rightarrow \Lambda M$$

of an open neighborhood of x in ΛM in terms of the exponential map; here compactness of M enters. The orthogonal splitting

$$X \simeq T_x W^u(x) \oplus T_x W^s(x) = X^- \oplus X^+$$

with associated orthogonal projections π_{\pm} is a key ingredient to make the analysis work. At this stage take the final identity as a definition. By a standard graph argument we assume without loss of generality that \mathcal{U} is of the form $W^u \times B^s$ where $W^u \subset X^-$ represents an open neighborhood of the critical point in its unstable manifold and $B^s \subset X^+$ is an open ball about 0. By ϕ we denote the local semiflow on \mathcal{U} which represents the heatflow under Φ ; see (5).

Hypothesis 1.1 (Local setup – figure 4). Fix a perturbation $V \in C^\infty(S^1 \times M)$ and a nondegenerate critical point x of \mathcal{S}_V of Morse index k and action c .

(a) Consider the coordinates on ΛM provided by Φ and modelled on the open subset $\mathcal{U} = W^u \times B^s$ of X . In these coordinates the origin $0 \in X$ represents x and $\mathcal{S} := \mathcal{S}_V \circ \Phi^{-1}$ represents the action. We denote closed balls about 0 by

$$\mathcal{B}_r := \{\|\cdot\|_X \leq r\}, \quad \mathcal{B}_r^+ := \{\|\cdot\|_{X^+} \leq r\}.$$

Choose the constant $\rho_0 > 0$ in the Lipschitz lemma 2.1 smaller, if necessary, such that $\mathcal{B}_{\rho_0} \subset \mathcal{U}$. Pick a sufficiently small constant $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0]$ the descending and ascending disks

$$W_\varepsilon^u(x) := W^u(x) \cap \{\mathcal{S}_V > c - \varepsilon\}, \quad W_\varepsilon^s(x) := W^s(x) \cap \{\mathcal{S}_V < c + \varepsilon\},$$

are contained in the coordinate patch $\Phi(\mathcal{B}_{\rho_0})$ and such that their closures are diffeomorphic to the closed unit disks in \mathbb{R}^k and X^+ , respectively. Existence of ε_0 follows by the Morse- and the Palais-Morse lemma.

(b) Fix $\mu \in (0, d)$ in the spectral gap (4) of the Jacobi operator. Pick $r \in (0, \rho_0)$ so small that $\mathcal{D} := S_\varepsilon^u \times \mathcal{B}_r^+$ is contained in \mathcal{B}_{ρ_0} and set $\mathcal{D}_\gamma := \{\gamma\} \times \mathcal{B}_r^+$.

(c) Our notation for objects expressed in coordinates will be the global notation with x omitted, for example W_ε^s and \mathcal{D}_γ .

Theorem 1.2 (Backward λ -lemma). *Assume the local setup of hypothesis 1.1. In particular, consider the hyperbolic fixed point 0 of the local semiflow ϕ on $\mathcal{U} \subset X$ given by (5) and the hypersurface $\mathcal{D} = S_\varepsilon^u \times \mathcal{B}_r^+ \subset \mathcal{B}_{\rho_0} \subset \mathcal{U}$ as illustrated by figure 2. Then the following is true. There is a ball \mathcal{B}^+ about zero in X^+ , a constant $T_0 > 0$, and a Lipschitz continuous map (defined by (24))*

$$\begin{aligned} \mathcal{G} : (T_0, \infty) \times S_\varepsilon^u \times \mathcal{B}^+ &\rightarrow W_\delta^u \times \mathcal{B}^+ \subset \mathcal{U} \\ (T, \gamma, z_+) &\mapsto (G_\gamma^T(z_+), z_+) =: \mathcal{G}_\gamma^T(z_+) \end{aligned}$$

of class C^1 . Each map $\mathcal{G}_\gamma^T : \mathcal{B}^+ \rightarrow X$ is bi-Lipschitz, a diffeomorphism onto its image, and $\mathcal{G}_\gamma^T(0) = \phi_{-T}\gamma =: \gamma_T$. The graph of \mathcal{G}_γ^T consists of those $z \in \mathcal{U}$ which satisfy $\pi_+ z \in \mathcal{B}^+$ and reach the fiber $\mathcal{D}_\gamma = \{\gamma\} \times \mathcal{B}_r^+$ at time T , that is

$$\mathcal{G}_\gamma^T(\mathcal{B}^+) = \phi_T^{-1} \mathcal{D}_\gamma \cap (X^- \times \mathcal{B}^+).$$

Furthermore, the graph map \mathcal{G}_γ^T converges uniformly, as $T \rightarrow \infty$, to the stable manifold graph map \mathcal{G}^∞ of theorem 2.6. More precisely, the estimate

$$\|\mathcal{G}_\gamma^T(z_+) - \mathcal{G}^\infty(z_+)\|_X \leq e^{-T\frac{\mu}{4}}$$

is true for all $T > T_0$, $z_+ \in \mathcal{B}^+$, and $\gamma \in S_\varepsilon^u$.

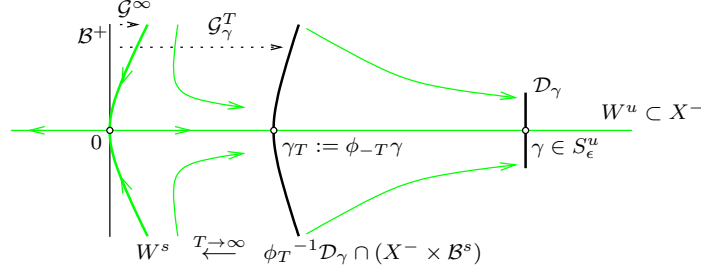


Figure 2: Backward λ -lemma

Theorem 1.3 (Uniform C^1 convergence). *Under the assumptions of theorem 1.2 each linearized graph map $d\mathcal{G}_\gamma^T(z_+) : X^+ \rightarrow X$ extends to a bounded linear operator on the L^2 completions and converges to $d\mathcal{G}^\infty(z_+)$, uniformly in $z_+ \in \mathcal{B}^+$. More precisely, assume $\gamma \in S_\varepsilon^u$, then*

$$\|d\mathcal{G}_\gamma^T(z_+)v\|_2 \leq 2\|v\|_2$$

and

$$\|d\mathcal{G}_\gamma^T(z_+)v - d\mathcal{G}^\infty(z_+)v\|_2 \leq 3ce^{-T\frac{\mu}{4}}\|v\|_2$$

for all $T > T_0$, $z_+ \in \mathcal{B}^+$, and v in the L^2 closure of X^+ .

Remark 1.4. (a) Theorem 1.2 recovers the common case of a single disk intersecting the unstable manifold transversely in one point γ near 0. Apply the implicit function theorem to bring the disk into the normal form $\{\gamma\} \times \mathcal{B}_r^+$. Check that \mathcal{G}_γ^T is defined without reference to any neighbors of γ . One could also throw in virtual disks to obtain the bundle $\mathcal{D} = S_\varepsilon^u \times \mathcal{B}_r^+$ in the hypothesis.

(b) Theorem 1.2 for $T = \infty$ recovers two known results. These appear as extreme cases concerning the radius r disk bundle $\mathcal{D} = S_\varepsilon^u \times \mathcal{B}_r^+$. 1) Disk of radius r sitting at 0: In this case $S_\varepsilon^u = \{0\}$, that is the disk bundle degenerates to just one disk sitting at the origin. This recovers the local stable manifold theorem 2.6. 2) Radius 0 disk bundle sitting at ∞ : This recovers the stable foliation constructed by [CL91]. Two points belong to the same leaf if under the semiflow φ_s their difference converges exponentially to zero, as $s \rightarrow \infty$.

Remark 1.5. (i) Pick $p \geq 1$ and restrict the action \mathcal{S}_V to the Banach manifold $W^{1,2p}(S^1, M)$. Then theorem 1.2 and theorem 1.3 continue to hold for an open subset \mathcal{U} of $X = W^{1,2p}(S^1, x^*TM)$ and the corresponding local semiflow. This refinement is used in the application in section 1.3; cf. remark 3.2.

(ii) All results in this paper extend to the more general class of perturbations satisfying axioms (V0-V3) in [SW03]; see [W].

(iii) The notation \mathcal{G}^∞ for the stable manifold graph map of theorem 2.6 is motivated by the observation that formally the preimage $\phi_T^{-1}(0)$ for $T = \infty$ corresponds to the local stable manifold $W^s(0, \mathcal{U})$ defined by (15).

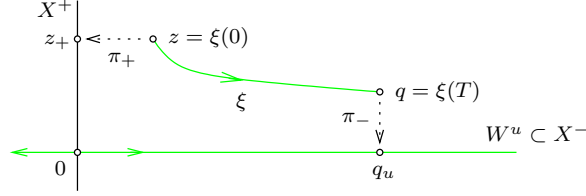


Figure 3: Data (T, q_u, z_+) for mixed Cauchy problem

(iv) The reason why we proved uniform C^1 convergence in theorem 1.3 with respect to the L^2 , rather than the $W^{1,2}$ norm, is that the fundamental step in section 1.3 involves the L^2 gradient property of the heat equation (1).

The following remark motivates why the map \mathcal{G} in theorem 1.2 should exist.

Remark 1.6 (Mixed Cauchy problem). Assume the hypotheses of the backward λ -lemma and fix $T > 0$. Each point z in the preimage $\phi_T^{-1}\mathcal{D}_\gamma$ corresponds to a unique semiflow line ξ such that $\xi(0) = z$ and ξ hits the fiber \mathcal{D}_γ precisely at time T , say in the point $q := \xi(T)$. Of course, we cannot change the order, i.e. first choosing an end point $q \in \mathcal{D}_\gamma$ and then determining a semiflow line ξ with $\xi(T) = q$. This would amount to solve the Cauchy problem for the heat equation in *backward* time, a problem well known to be ill defined in general: Indeed any nonsmooth element $q \in \mathcal{D}_\gamma \subset W^{1,2}$ cannot be reached, since the point $\xi(T)$ on any heatflow trajectory ξ is necessarily smooth – due to the strongly regularizing effect of the heatflow for $T > 0$; see e.g. [W]. However, consider the splitting $X = X^- \oplus X^+$ in unstable and stable tangent spaces. In section 2.3 we will see that each element of X^- is smooth. So specifying only the X^- part of the endpoint does at least not contradict regularity. The key idea is to introduce the notion of a **mixed Cauchy problem**: Apart from time T only the stable part z_+ of the initial point is prescribed – in exchange of prescribing in addition the unstable part q_u of the end point; see figure 3. Indeed the representation formula in proposition 2.5 shows that the mixed Cauchy problem is equivalent to the usual Cauchy problem with initial value z . Because the latter admits a unique solution, so does the mixed Cauchy problem.

1.3 Application

The method in [W12] to construct a Morse filtration of the loop space is inspired by Conley theory [Co78]. For reals $\varepsilon, \tau > 0$ denote by N the path connected component of x of the open set $\{\gamma \in \Lambda M \mid \mathcal{S}_V(\gamma) < c + \varepsilon, \mathcal{S}_V(\varphi_\tau \gamma) > c - \varepsilon\}$. The open subset $L := \{\gamma \in N \mid \mathcal{S}_V(\varphi_\tau \gamma) > c - \varepsilon\}$ of N is called an **exit set** of N and (N, L) is called a **Conley pair** for the semiflow invariant set $\{x\}$. A fundamental step is to prove that relative singular homology is given by

$$H_\ell(N, L) \simeq \begin{cases} \mathbb{Z} & , \ell = k := \text{ind}(x), \\ 0 & , \text{otherwise.} \end{cases}$$

The part of N in the unstable manifold is an open k disk and $L \cap W^u(x)$ is an “outer” annulus thereof. So the relative homology of these parts has the required property and it suffices to show that $(N \cap W^u(x), L \cap W^u(x))$ is a deformation retract of (N, L) . Observe then that N contains the ascending disk $W_\varepsilon^s(x)$, from now on abbreviated by W_ε^s , whose part in the unstable manifold is precisely the critical point x itself. In this case the semiflow φ_s itself provides the desired deformation. Obviously this fails on the complement of the ascending disk. Now the backward λ -lemma comes in. It endows (N, L) , as we show in [W12], with the structure of a codimension k foliation whose leaves $N(\gamma_u)$ are parametrized by $\gamma_u \in N \cap W^u(x)$. Furthermore, each leaf is diffeomorphic to a neighborhood U_{γ_u} of W_ε^s in $W^s(x)$ and $N(x) = W_\varepsilon^s = U_x(W_\varepsilon^s)$. These diffeomorphisms

$$\Psi_{\gamma_u} : U_{\gamma_u}(W_\varepsilon^s) \xrightarrow{\cong} N(\gamma_u), \quad \Psi_{\gamma_u}(x) = \gamma_u, \quad \Psi_0 = id_{W_\varepsilon^s},$$

allow to extend the desirable deformation property provided by φ_s on W_ε^s to all of N . Indeed pick $\gamma \in N$. Then by the foliation property γ lies on a leaf, say on $N(\gamma_u)$, and the map

$$\theta_s(\gamma) := \Psi_{\gamma_u} \circ \varphi_s \circ \Psi_{\gamma_u}^{-1}(\gamma), \quad s \in [0, \infty],$$

deforms N onto its part in the unstable manifold. So we are done. Well, note the subtlety arising due to the deformation having to take place entirely in N which is equivalent to invariance of $U_{\gamma_u}(W_\varepsilon^s)$ under φ_s . For $U_x(W_\varepsilon^s) = W_\varepsilon^s$ this follows immediately from the fact that the action decreases along the heatflow. Since $\dim U_{\gamma_u}(W_\varepsilon^s) = \infty$, the general case is nontrivial. Apart from the Palais-Smale condition, the analytic properties of the graph maps \mathcal{G}_γ^T provided by theorems 1.2 and 1.3 enter heavily. For details we refer to [W12].

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2 Toolbox

Throughout fix a nondegenerate critical point x of \mathcal{S}_V and a constant $p \geq 1$. Assume that \mathcal{S}_V is defined on the Banach manifold $\Lambda^{1,2p}M := W^{1,2p}(S^1, M)$. The critical points do not depend on p by C^∞ regularity and the tangent space at x is given by

$$X = W^{1,2p}(S^1, x^*TM).$$

Representing the Hessian of \mathcal{S}_V at x with respect to the L^2 inner product on the loop space gives rise to the **Jacobi operator** A_x given by

$$A_x \xi = -\nabla_t \nabla_t \xi - R(\xi, \dot{x})\dot{x} - \nabla_\xi \nabla V_t(x) \quad (2)$$

for every smooth vector field ξ along the loop x . Here R denotes the Riemannian curvature tensor. Viewed as an unbounded operator on a general Sobolev space

$$W^{k,q} := W^{k,q}(S^1, x^*TM)$$

with dense domain $W^{k+2,q}$, where $k \in \mathbb{N}_0$ and $q \geq 1$, the spectrum of A_x does not depend on (k, q) and takes the form of a sequence of real eigenvalues (counting multiplicities)

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k < 0 < \lambda_{k+1} \leq \lambda_{k+2} \leq \dots \quad (3)$$

which converges to ∞ . Calculation of the spectrum is standard. One picks the Hilbert space case $(k, q) = (0, 2)$ and proves first that A_x admits a compact self-adjoint resolvent. In the second step it remains to prove C^∞ regularity of eigenfunctions. The **spectral gap** $(0, d)$ of A_x is determined by

$$d := \text{dist}(0, \sigma(A)) = \min\{-\lambda_k, \lambda_{k+1}\} > 0. \quad (4)$$

By σ_\pm we denote the **positive** and **negative part of the spectrum of A_x** . The critical point x is called **nondegenerate** if zero is not in the spectrum. Equivalently x is a **hyperbolic** fixed point of φ_s whenever $s > 0$, that is the spectrum of the linearized flow $d\varphi_s(x)$ does not contain 1. The **Morse index** of x is the number k of negative eigenvalues of A_x counted with multiplicities.

It is worthwhile to mention some of the useful properties enjoyed by the action functional: It strictly decreases along nonconstant heatflow trajectories. It is bounded below and satisfies the Palais-Smale condition.

In the following subsections we provide the analytical tools required in the proof of the backward λ -lemma. They are all well known, surely by the experts, and so we simply list them without proofs. On the other hand, some are difficult to find in the literature, e.g. sectoriality of A_x in the relevant periodic case. So here is some good news for non-experts:

Convention. The proof of any assertion attributed **well known** in section 2 is given in [W]. The same holds for facts stated **without reference**.

2.1 Local semiflow

Any path $[0, T] \ni s \mapsto u_s$ in a sufficiently small neighborhood $\mathcal{U}(x) = \Phi(\mathcal{U})$ of x in the Banach manifold $\Lambda^{1,2p}M$ corresponds to a path $\zeta : [0, T] \rightarrow \mathcal{U}$, $s \mapsto \zeta(s)$, determined uniquely by the identity $u_s = \exp_x \zeta(s) =: \Phi(\zeta(s))$ pointwise for $t \in S^1$. Applying the operators ∂_s and $\nabla_t \nabla_t$ to this identity transforms the Cauchy problem on ΛM associated to (1) into the equivalent Cauchy problem

$$\frac{d}{ds} \zeta(s) + A_x \zeta(s) = f(\zeta(s)), \quad \zeta(0) = z := \Phi^{-1}(\gamma) \in \mathcal{U}, \quad (5)$$

for maps $\zeta : [0, T] \rightarrow \mathcal{U}$ into the Banach space $X = W^{1,2p}(S^1, x^*TM)$ where T depends on $z \in \mathcal{U}$. The solution to (5) defines the local semiflow $\phi_s z := \zeta(s)$ on

\mathcal{U} and A_x denotes the Jacobi operator (2) on $W^{1,2p}$ with dense domain $W^{3,2p}$. The nonlinearity $f : W^{1,2p} \rightarrow L^p$ is given pointwise at (s, t) by the identity

$$\begin{aligned} f(\zeta) = & E_2(x, \zeta)^{-1} \left(E_{11}(x, \zeta) (\partial_t x, \partial_t x) + 2E_{12}(x, \zeta) (\partial_t x, \nabla_t \zeta) \right. \\ & + E_{22}(x, \zeta) (\nabla_t \zeta, \nabla_t \zeta) + \nabla V_t(\exp_x \zeta) - E_1(x, \zeta) \nabla V_t(x) \Big) \\ & - R(\zeta, \dot{x}) \dot{x} - \nabla_\zeta \nabla V_t(x). \end{aligned} \quad (6)$$

To arrive at this form of f we used the well known covariant partial derivatives of the exponential map, that is the multilinear maps

$$E_{i_1 \dots i_j}(q, v) : (T_q M)^{\times j} \rightarrow T_{\exp_q v} M$$

depending smoothly on $(q, v) \in TM$ for each $j \in \mathbb{N}$. They satisfy the identities

$$E_1(q, 0) = E_2(q, 0) = \mathbb{1}, \quad E_{11}(q, 0) = E_{21}(q, 0) = E_{22}(q, 0) = 0, \quad (7)$$

and admit the symmetries $E_{12}(q, z)(v, w) = E_{21}(q, z)(w, v)$, $E_{22}(q, z)(v, w) = E_{22}(q, z)(w, v)$, and $E_{11}(q, z)(v, w) - E_{11}(q, z)(w, v) = E_2(q, z)R(v, w)z$ for all $q \in M$ and $z, v, w \in T_q M$. Furthermore, if $\gamma : \mathbb{R} \rightarrow M$ is a smooth curve in M and ξ is a smooth vector field along γ , then

$$E_{112}(\gamma, 0)(\dot{\gamma}, \dot{\gamma}, \xi) = R(\xi, \dot{\gamma})\dot{\gamma} \quad (8)$$

pointwise for every $t \in \mathbb{R}$.

So after all what is the advantage of reformulating the Cauchy problem? Obviously the linear structure of X to start with. However, the really great features are that the spectral splitting $X \simeq X^- \oplus X^+$ induced by A_x is preserved by the semigroups of section 2.3 and that the part X^- is of finite dimension k , consists of smooth elements, and as a set does not depend on p .

2.2 Lipschitz estimate for the nonlinearity

Lemma 2.1 (Locally Lipschitz). *Fix a constant $p \geq 1$. Then there are constants $\rho_0, \kappa_* > 0$ and a continuous nondecreasing function κ on the interval $[0, \rho_0]$ with $\kappa(0) = 0$, all depending on p , such that the following is true. The nonlinearity $f : W^{1,2p} \supset \mathcal{B}_{\rho_0} \rightarrow L^p$ given by (6) is of class C^1 and satisfies the identities $f(0) = 0$ and $df(0) = 0$ and the estimates*

$$\begin{aligned} \|f(\xi) - f(\eta)\|_p &\leq \kappa(\rho) \|\xi - \eta\|_{1,2p}, \\ \|df(\xi)v - df(\eta)v\|_p &\leq \kappa_* \|\xi - \eta\|_{1,2p} \|v\|_{1,2p}, \end{aligned}$$

whenever $\|\xi\|_{1,2p}, \|\eta\|_{1,2p} \leq \rho < \rho_0$ and $v \in W^{1,2p}$.

Corollary 2.2. *Assume lemma 2.1. Then $\|df(\xi)v\|_p \leq \kappa(\rho)\|v\|_{1,2p}$ whenever $\|\xi\|_{1,2p} \leq \rho < \rho_0$ and $v \in W^{1,2p}$.*

Proof. Use that $df(\xi)v = \lim_{\tau \rightarrow 0} \frac{f(\xi + \tau v) - f(\xi)}{\tau}$ and apply lemma 2.1. \square

Remark 2.3. (a) That $\kappa(\rho)$ is nondecreasing in ρ is used to prove the assertion of theorem 2.6 that at the fixed point 0 the stable manifold is tangent to X^+ .
(b) The Lipschitz estimate for df with constant κ_* is required, firstly, to prove that the graph map \mathcal{G}_γ^T is of class C^1 in T and, secondly, to prove uniform convergence of its derivative to the derivative of the stable manifold graph map, as $T \rightarrow \infty$; see proof of theorem 1.2 step 4 and proof of theorem 1.3 step II.

Proof of the Lipschitz lemma 2.1. Fix $\rho_1 > 0$ sufficiently small such that $\mathcal{B}_{\rho_1} \subset \mathcal{U}$. Consider the Sobolev embedding $W^{1,2p} \hookrightarrow L^\infty$ with Sobolev constant c_p and the injectivity radius $\iota > 0$ of the closed Riemannian manifold M . Setting $\rho_0 := \min\{\rho_1, \iota/8c_p\}$ guarantees that whenever $\|\xi\|_{1,2p} \leq \rho < \rho_0$, then $\xi \in \mathcal{U}$ and $\|\xi\|_\infty \leq c_p \rho \leq \frac{\iota}{8}$. Observe that ξ takes values in the subset $\mathcal{O}_{\rho/3} \subset TM$ which by definition consists of all pairs (q, v) such that $q \in M$ and $v \in T_q M$ satisfies $|v| \leq c_p \rho$. Note that \mathcal{O}_0 is the zero section.

From now on assume $\rho \in [0, \rho_0]$ and $\|\xi\|_{1,2p}, \|\eta\|_{1,2p} \leq \rho$. To see that $f(0) = 0$ use (6) and (7). Use in addition (8) to prove that $df(0)\zeta := \frac{d}{d\tau}f(\tau\zeta) = 0$. Abbreviate $X := \eta - \xi$ to obtain that

$$\begin{aligned} f(\xi) - f(\eta) &= (E_2(x, \xi)^{-1}E_{11}(x, \xi) - E_2(x, \eta)^{-1}E_{11}(x, \eta))(\dot{x}, \dot{x}) + R(X, \dot{x})\dot{x} \\ &\quad + 2(E_2(x, \xi)^{-1}E_{21}(x, \xi)\nabla_t \xi - E_2(x, \eta)^{-1}E_{21}(x, \eta)\nabla_t \eta) \dot{x} \\ &\quad + E_2(x, \xi)^{-1}E_{22}(x, \xi)(\nabla_t \xi, \nabla_t \xi) - E_2(x, \eta)^{-1}E_{22}(x, \eta)(\nabla_t \eta, \nabla_t \eta) \\ &\quad + E_2(x, \xi)^{-1}\nabla V_t(\exp_x \xi) - E_2(x, \eta)^{-1}\nabla V_t(\exp_x \eta) + \nabla_X \nabla V_t(x) \\ &\quad - (E_2(x, \xi)^{-1}E_1(x, \xi) - E_2(x, \eta)^{-1}E_1(x, \eta))\nabla V_t(x) \end{aligned}$$

pointwise at every $t \in S^1$. We denote the last five lines of the formula above by I through V , respectively, and deal with each one separately. For now think of ξ as a fixed parameter and view $\eta(X) = \xi + X$ as a function of X . Then each line becomes a (smooth) function of $X(t)$ depending on additional quantities such as certain derivatives of ξ , X , and x all evaluated at t . For instance, term I becomes the identity

$$I(X) = (E_2(x, \xi)^{-1}E_{11}(x, \xi) - E_2(x, \eta(X))^{-1}E_{11}(x, \eta(X)))(\dot{x}, \dot{x}) + R(X, \dot{x})\dot{x}$$

pointwise at every $t \in S^1$. By straightforward calculation

$$\begin{aligned} dI(X)Y &= \frac{D}{d\tau}\Big|_{\tau=0} I(X + \tau Y) \\ &= (E_2(x, \eta(X))^{-1}E_{22}(x, \eta(X)))(E_2(x, \eta(X))^{-1}E_{11}(x, \eta(X))(\dot{x}, \dot{x}), Y) \\ &\quad - E_2(x, \eta(X))^{-1}E_{112}(x, \eta(X))(\dot{x}, \dot{x}, Y) + R(Y, \dot{x})\dot{x} \end{aligned}$$

pointwise at every $t \in S^1$. Note that $\eta(\sigma X) = \sigma\eta + (1 - \sigma)\xi$, for $\sigma \in [0, 1]$ and pointwise in t , takes values in $\mathcal{O}_{2\rho/3} \subset \mathcal{O}_\rho \subset \mathcal{O}_{\rho_0}$. Note further that $I(0) = 0$. Hence by Taylor's theorem there is a constant $\sigma = \sigma(t) \in [0, 1]$ such that

$$\begin{aligned} |I(X)| &= |dI(\sigma X)X| \leq \|E_2^{-1}\|_{L^\infty(\mathcal{O}_{\rho_0})}^2 \|E_{22}\|_{L^\infty(\mathcal{O}_\rho)} \|E_{11}\|_{L^\infty(\mathcal{O}_\rho)} |\dot{x}|^2 |X| \\ &\quad + \|E_2^{-1}E_{112}(*, *, \cdot) - R(\cdot, *)\|_{L^\infty(\mathcal{O}_\rho)} |\dot{x}|^2 |X| \\ &=: \kappa_1(\rho) |\dot{x}|^2 |X| \end{aligned}$$

pointwise at every $t \in S^1$. Note that the function κ_1 depends continuously on $\rho \in [0, \rho_0]$ and that $\kappa_1(0) = 0$ using the curvature identity (8) and because $E_{ij}(\cdot, 0) = 0$ for $i, j \in \{1, 2\}$ by (7). Now pull out sup norms and apply the Hölder inequality $\|gh\|_p \leq \|g\|_{2p}\|h\|_{2p}$ to obtain the desired Lipschitz estimate for term one, namely $\|I(X)\|_p \leq \kappa_1(\rho)\|\dot{x}\|_{2p}^2\|X\|_\infty \leq c_p\kappa_1(\rho)\|\dot{x}\|_{2p}^2\|\xi - \eta\|_{1,2p}$. Since critical points are automatically smooth and the domain S^1 is compact there is actually a C^1 bound for x . In the case $p = 1$ we may actually use the definition of the action to obtain $\|I(X)\|_1 \leq 2c_1(\mathcal{S}_V(x) + \|V\|_\infty)\kappa_1(\rho)\|\xi - \eta\|_{1,2}$. The argument for terms two through five is analogous; see [W] for details.

To see that f is of class C^1 observe that $df(\xi)X = dI(0)X + \dots + dV(0)X$. Careful inspection term by term then shows that each of the five terms in this sum depends continuously on ξ with respect to the $W^{1,2p}$ topology.

It remains to prove the second Lipschitz estimate, that is the one for the difference of derivatives $df(\xi) - df(\eta)$. Unfortunately, the number of terms appearing during the calculation is rather large. Fortunately, we are only claiming existence of a constant κ_* . Straightforward calculation shows that

$$\begin{aligned} df(\xi)v &= -R(v, \dot{x})\dot{x} - E_2(x, \xi)^{-1} \left[E_{22}(x, \xi) (E_2(x, \xi)^{-1} E_{11}(x, \xi) (\dot{x}, \dot{x}), v) \right. \\ &\quad - E_{22}(x, \xi) (E_2(x, \xi)^{-1} [2E_{12}(x, \xi) (\dot{x}, \nabla_t \xi) + E_{22}(x, \xi) (\nabla_t \xi, \nabla_t \xi)], v) \\ &\quad - E_{22}(x, \xi) (E_2(x, \xi)^{-1} \nabla V_t(\exp_x \xi), v) + E_{22}(x, \xi) (\nabla V_t(x), v) \\ &\quad + E_{112}(x, \xi) (\dot{x}, \dot{x}, v) + 2E_{122}(x, \xi) (\dot{x}, \nabla_t \xi, v) + 2E_{12}(x, \xi) (\dot{x}, \nabla_t v) \\ &\quad + E_{222}(x, \xi) (\nabla_t \xi, \nabla_t \xi, v) + 2E_{22}(x, \xi) (\nabla_t \xi, \nabla_t v) \\ &\quad \left. + \frac{D}{d\tau} \Big|_{\tau=0} \nabla V_t(\exp_x(\xi + \tau v)) - E_{12}(x, \xi) (\nabla V_t(x), v) \right] - \nabla_v \nabla V_t(x). \end{aligned}$$

Denote the fourteen terms in this sum by $\sum_{j=1}^{14} H_j(\xi)v$. For $X := \eta - \xi$ set $2F_j(X)v := H_j(\xi)v - H_j(\xi + X)v$. For instance, consider F_8 . We get that

$$\begin{aligned} dF_8(X)(v, Y) &= \frac{D}{d\tau} \Big|_{\tau=0} [E_2(x, \xi + X + \tau Y)^{-1} E_{122}(x, \xi + X + \tau Y) (\dot{x}, \nabla_t (\xi + X + \tau Y), v)] \\ &= -E_2^{-1} E_{22} (E_2^{-1} E_{122} (\dot{x}, \nabla_t \xi + \nabla_t X, v), Y) \\ &\quad + E_2^{-1} E_{1222} (\dot{x}, \nabla_t \xi + \nabla_t X, v, Y) + E_2^{-1} E_{122} (\dot{x}, \nabla_t Y, v) \end{aligned}$$

where the maps are evaluated at $(x, \xi + X)$. Since $F_8(0) = 0$ there is by Taylor's theorem, pointwise at every $t \in S^1$, a constant $\sigma = \sigma(t) \in [0, 1]$ such that

$$\begin{aligned} \|2F_8(X)v\|_p &= \|2dF_8(\sigma X)(v, X)\|_p \\ &\leq \|E_2^{-1}\|_{L_{\rho_0}^\infty} \left(\|E_2^{-1}\|_{L_{\rho_0}^\infty} \|E_{22}\|_{L_\rho^\infty} \|E_{122}\|_{L_{\rho_0}^\infty} + \|E_{1222}\|_{L_{\rho_0}^\infty} \right) \\ &\quad \cdot \|\dot{x}\|_{2p} \left(\sigma \|\eta\|_{1,2p} + (1 - \sigma) \|\xi\|_{1,2p} \right) \|v\|_\infty \|\xi - \eta\|_\infty \\ &\quad + \|E_2^{-1}\|_{L_{\rho_0}^\infty} \|E_{122}\|_{L_{\rho_0}^\infty} \|\dot{x}\|_{2p} \|\xi - \eta\|_{1,2p} \|v\|_\infty \end{aligned}$$

where we abbreviated $L_\rho^\infty := L^\infty(\mathcal{O}_\rho)$. This proves the Lipschitz estimate for term eight. Note that $F_1 \equiv 0 \equiv F_{14}$. The estimates for the other eleven F -terms follow similarly. This completes the proof of the Lipschitz lemma 2.1. \square

2.3 Semigroups and splitting

For any $p \in [1, \infty)$ and $k \in \mathbb{N}_0$ the negative Jacobi operator $-A := -A_x$ on $Z := W^{k,p}$ with dense domain $W^{k+2,p}$ and given by (2) is sectorial and therefore generates the strongly continuous semigroup $e^{-sA} \in \mathcal{L}(Z)$ given by

$$e^{sA} := \frac{1}{2\pi i} \int_{\gamma} e^{s\lambda} R(\lambda, -A) d\lambda, \quad \forall s > 0, \quad (9)$$

and by $e^{0A} := \mathbb{1}_Z$ for $s = 0$. Here R denotes the resolvent and $\gamma : \mathbb{R} \rightarrow \mathbb{C}$ is a suitable path inside the resolvent set $\rho(-A)$. Sectoriality of $-A$ is well known, but a proof for the periodic domain S^1 is hard to find, unlike for the domain \mathbb{R} . So we provided the details in [W]. By nondegeneracy of x the operator $-A$ is **hyperbolic**, that is its spectrum and the imaginary axis $i\mathbb{R}$ are disjoint. Pick a counter-clockwise oriented closed circle $\gamma^+ : S^1 \rightarrow (0, \infty) \times i\mathbb{R}$ which encloses the positive part $\{-\lambda_k, \dots, -\lambda_1\}$ of the spectrum of $-A$. The linear operators

$$\pi_- := \frac{1}{2\pi i} \int_{\gamma^+} R(\lambda, -A) d\lambda, \quad \pi_+ := \mathbb{1} - \pi_-, \quad (10)$$

are elements of $\mathcal{L}(Z)$ called **spectral projections**, because $(\pi_{\pm})^2 = \pi_{\pm}$.

We collect key facts of semigroup theory. By boundedness of π_{\pm} the images

$$Z^{\pm} := \text{range } \pi_{\pm}, \quad (11)$$

are closed (Banach) subspaces. As a vector space Z^- is spanned by k eigenfunctions corresponding to the $k = \text{ind}(x)$ negative eigenvalues of A . In particular $Z^- \subset C^\infty(S^1, x^*TM)$. In contrast Z^+ is the $W^{k,p}$ closure of the sum of eigenspaces corresponding to positive eigenvectors of A . Thus $Z^+ = Z^+(k, p)$. The obvious identity $\pi_- \pi_+ = \pi_+ \pi_- = 0$ shows that $Z \simeq Z^- \oplus Z^+$. Moreover, this **splitting** is preserved by A and the restrictions of A to the Banach subspaces Z^{\pm} are denoted by A^{\pm} . Since the semigroup e^{-sA} preserves both subspaces Z^{\pm} , the restrictions $e^{-sA}|_{Z^{\pm}}$ are semigroups as well. They are called **subspace semigroups**. On the other hand, the restrictions $-A^{\pm}$ themselves are sectorial operators on the Banach spaces Z^{\pm} with dense domains $Z^{\pm} \cap D(A)$. Therefore they generate strongly continuous semigroups $e^{-sA^{\pm}}$ on Z^{\pm} . But these coincide with the subspace semigroups due to the resolvent identity $R(\lambda, -A)|_{Z^{\pm}} = R(\lambda, -A^{\pm})$ which holds for every λ in the resolvent set $\rho(-A) \subset \rho(-A^{\pm})$. The upshot is the formula

$$e^{-sA} = e^{-sA^-} \oplus e^{-sA^+}, \quad s \geq 0.$$

Note that $D(A) \cap Z^- = Z^-$ by smoothness. Thus $A^- \in \mathcal{L}(Z^-)$ and the series

$$e^{-sA^-} := \sum_{k=0}^{\infty} \frac{(-sA^-)^k}{k!}, \quad \forall s \in \mathbb{R}, \quad (12)$$

is well defined providing a norm continuous group which for $s \geq 0$ coincides with the subspace semigroup $e^{-sA}|_{Z^-}$. For *negative* times $s \leq 0$ it decays exponentially $\|e^{-sA^-}\|_{\mathcal{L}(Z^-)} \leq ce^{-s\lambda_k} \leq ce^{s\mu}$. The constructions above commute

with Sobolev embeddings $W^{\ell,q} \hookrightarrow W^{k,p}$. For π_{\pm} this is again a consequence of a resolvent identity.

Proposition 2.4. *Fix integers $\ell \geq k \geq 0$ and constants $q \geq p \geq 1$. Consider the Jacobi operator $A := A_x$ on $Z := W^{k,p}$ with dense domain $W^{k+2,p}$ and its restrictions A^{\pm} to the closed subspaces $Z^{\pm} := \pi_{\pm}(Z)$. Fix $\mu > 0$ in the spectral gap (4) of A . Then there is a constant $c = c(\ell, k, q, p, \mu)$ such that*

- (a) *The operator $-A$ on L^1 generates the strongly continuous semigroup e^{-sA} on Y given by (9). Both commute with the spectral projections π_{\pm} in (10).*
- (b) *The subspace semigroup $e^{-sA}|$ on Z coincides with the strongly continuous semigroup $e^{-sA|}$ generated by the restriction of $-A$ to Z . Subspace semigroup and $A|$ commute with the spectral projections $\pi_{\pm}(-A|) = \pi_{\pm}(-A)|$.*
- (c) *The restriction of $-A$ to Z^- generates the norm continuous group e^{-sA^-} on Z^- given by the exponential series (12). For positive times this group is equal to the subspace semigroup $e^{-sA}|$. For negative times it holds that*

$$\|e^{-sA^-} \pi_{-}\|_{\mathcal{L}(W^{k,p}, W^{\ell,q})} + \|e^{-sA^-} \pi_{-}\|_{\mathcal{L}(Z)} \leq ce^{s\mu}, \quad s \leq 0.$$

- (d) *Restricting e^{-sA} to Z^+ gives a strongly continuous semigroup on Z^+ and*

$$\|e^{-sA} \pi_{+}\|_{\mathcal{L}(W^{k,p}, W^{\ell,q})} \leq cs^{-\frac{1}{2}(\frac{1}{p} - \frac{1}{q} + \ell - k)} e^{-s\mu}, \quad s > 0. \quad (13)$$

2.4 The representation formula

Proposition 2.5. *Fix a constant $p \geq 1$ and abbreviate $X = W^{1,2p}$ and $Y = L^p$. Consider the nonlinearity $f : X \supset \mathcal{U} \rightarrow Y$ given by (6) and the constant ρ_0 provided by the Lipschitz lemma 2.1. Pick $T > 0$ and assume $\xi : [0, T] \rightarrow X$ is a map bounded by ρ_0 thus taking values in \mathcal{U} . Then the following are equivalent.*

- (a) *The map $\xi : [0, T] \rightarrow Y$ is the (unique) solution of the Cauchy problem (5) with initial value $\xi(0)$.*
- (b) *The map $\xi : (0, T] \rightarrow X$ is continuous¹ and satisfies the integral equation, also called **representation formula**, given by*

$$\begin{aligned} \xi(s) &= e^{-sA} \pi_{+} \xi(0) + \int_0^s e^{-(s-\sigma)A} \pi_{+} f(\xi(\sigma)) d\sigma \\ &\quad + e^{-(s-T)A^-} \pi_{-} \xi(T) - \int_s^T e^{-(s-\sigma)A^-} \pi_{-} f(\xi(\sigma)) d\sigma \end{aligned} \quad (14)$$

for every $s \in [0, T]$. In the limit $T \rightarrow \infty$ the third term disappears.

¹hence $f \circ \xi : (0, T] \rightarrow Y$ is continuous and, by the Lipschitz lemma 2.1, bounded.

2.5 Local stable manifold theorem

Theorem 2.6 (C^1 Lipschitz graph). *Fix a constant $p \geq 1$. Assume the local setup of hypothesis 1.1. In particular, consider the hyperbolic fixed point 0 of the local semiflow ϕ on $\mathcal{U} \subset X = W^{1,2p}(S^1, x^*TM)$; cf. figure 4. Then the following is true. There is a closed ball $\mathcal{B}^+ \subset X^+$ of radius $r > 0$ about 0 such that a neighborhood of 0 in the **local stable manifold***

$$W^s(0, \mathcal{U}) := \left\{ z \in \mathcal{U} \mid \phi(s, z) \in \mathcal{U} \ \forall s > 0 \text{ and } \lim_{s \rightarrow \infty} \phi(s, z) = 0 \right\}. \quad (15)$$

is a Lipschitz graph over \mathcal{B}^+ , tangent to X^+ at 0. More precisely, there is a Lipschitz continuous C^1 map

$$\mathcal{G}^\infty = (G, id) : \mathcal{B}^+ \rightarrow X^- \times \mathcal{B}^+, \quad G(0) = 0, \quad dG(0) = 0,$$

such that $\mathcal{G}^\infty(\mathcal{B}^+) = \text{graph } G$ is a neighborhood of 0 in $W^s(0, \mathcal{U})$; cf. figure 2.

Proposition 2.7 (L^2 extension). *Under the assumptions of theorem 2.6 the linearization $d\mathcal{G}^\infty(z_+) : X^+ \rightarrow X$ extends to a bounded linear operator on the L^2 completions, uniformly in $z_+ \in \mathcal{B}^+$. More precisely, it holds that*

$$\|d\mathcal{G}^\infty(z_+)v\|_2 \leq 2\|v\|_2, \quad \|d\mathcal{G}^\infty(z_+)v - v\|_2 \leq \frac{1}{4}\|v\|_2,$$

for all $v \in \pi_+(L^2)$ and $z_+ \in \mathcal{B}^+$.

The local stable manifold theorem 2.6 is well known; see e.g. Henry's book [He81, thm. 5.2.1] for a proof by the contraction method. In finite dimensions the theorem is also called Hadamard-Perron theorem [Hd01, Pe28]. Observe that proofs of theorem 2.6 and proposition 2.7 arise as simple special cases of the proofs in section 3, formally set $T = \infty$. Now we recall the **contraction method** for the stable manifold theorem. Pick a value for each parameter of interest, in our case $z_+ \in X^+$. Our object of interest is a heatflow line $\eta : [0, \infty) \rightarrow \mathcal{U}$ whose initial value projects to z_+ under π_+ and which converges to the origin, as $s \rightarrow \infty$. Then find a complete metric space, namely

$$Z = Z_\mu^\rho := \left\{ \eta \in C^0([0, \infty), X) \mid \|\eta\|_{\text{exp}} := \sup_{s \geq 0} e^{s\frac{\mu}{2}} \|\eta(s)\|_X \leq \rho \right\} \quad (16)$$

for suitable constants ρ and μ , and a strict contraction on Z , namely

$$(\Psi_{z_+} \eta)(s) := e^{-sA} z_+ + \int_0^s e^{-(s-\sigma)A} \pi_+ f(\eta(\sigma)) d\sigma - \int_s^\infty e^{-(s-\sigma)A^-} \pi_- f(\eta(\sigma)) d\sigma,$$

such that the (unique) fixed point η_{z_+} is the initial object of interest. Use the representation formula (14) for $T = \infty$ to see that this is indeed true. In fact this setup works for any μ in the spectral gap (4) of the Jacobi operator $A := A_x$ and whenever $\rho > 0$ is sufficiently small, that is if (19) holds. The map

$$\begin{aligned} G : \mathcal{B}^+ &\rightarrow X^-, & \mathcal{B}^+ &:= \{z_+ \in X^+ : \|z_+\|_X \leq \rho/2c\}, \\ z_+ &\mapsto \pi_- (\eta_{z_+}(0)) \end{aligned}$$

has the properties asserted by theorem 2.6; here c is given by proposition 2.4.

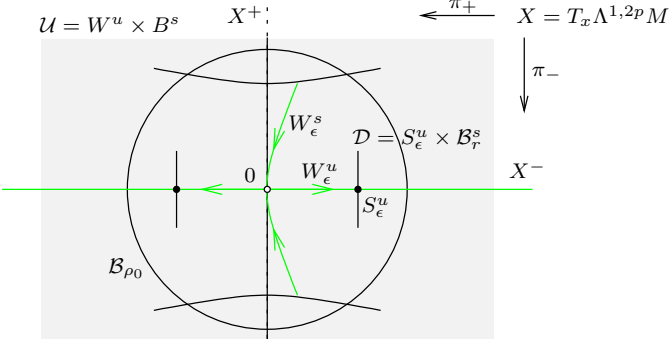


Figure 4: Local setup

Remark 2.8 (Unstable manifold). The contraction method also serves ² to represent the elements of the **local unstable manifold** $W^u(0, \mathcal{U})$ which by definition is the set of end points of all heatflow trajectories $\tilde{\eta}$ in \mathcal{U} which are parametrized by $(-\infty, 0]$ and emanate at time $-\infty$ from 0. Given z_- in some ball $\mathcal{B}^- = \mathcal{B}^-(\rho) \subset X^-$, then the heatflow trajectory $\tilde{\eta}$ emanating from 0 and satisfying $\pi_- \tilde{\eta}(0) = z_-$ is the unique fixed point $\tilde{\eta}_{z_-}$ of the map defined by

$$\begin{aligned} (\Phi_{z_-} \tilde{\eta})(s) &:= e^{-sA^-} z_- - \int_s^0 e^{-(s-\sigma)A^-} \pi_- f(\tilde{\eta}(\sigma)) d\sigma \\ &\quad + \int_{-\infty}^s e^{-(s-\sigma)A} \pi_+ f(\tilde{\eta}(\sigma)) d\sigma \end{aligned}$$

for every $s \leq 0$ which acts as a strict contraction on the complete metric space

$$\tilde{Z} = \tilde{Z}_\mu^\rho := \{ \tilde{\eta} \in C^0((-\infty, 0], X) \mid \|\tilde{\eta}\|_{\text{exp}} := \sup_{s \leq 0} e^{-s\frac{\mu}{2}} \|\tilde{\eta}(s)\|_X \leq \rho \}$$

whenever $\rho > 0$ is sufficiently small.

3 Proofs

3.1 Proof of the backward λ -lemma (theorem 1.2)

Uniform exponential convergence in step 6 is the heart of the proof. It relies on a suitable time decomposition of trajectories. Throughout assume

Hypothesis 3.1. Assume the local setup of hypothesis 1.1. In particular, pick a constant $p \geq 1$ and coordinates modelled on $\mathcal{U} \subset X$; see figure 4. Consider the constants $\rho_0 > 0$ and $\kappa_* \geq 1$ and the continuous function $\kappa(\rho)$ with $\kappa(0) = 0$ provided by the Lipschitz lemma 2.1 for this choice of \mathcal{U} . Fix μ in the spectral

²see e.g. [He81, thm. 5.2.1, proof of thm. 5.1.3]

gap $(0, d)$ given by (4) and a constant $c \geq 1$ satisfying proposition 2.4 for finitely many relevant choices of (ℓ, k, q, p) . Without loss of generality we assume that ³

$$c^2 \rho_0 \kappa_* \left(\frac{9}{\mu^{\frac{1}{4}}} + \frac{4}{3\mu} + 4 \right) \leq \frac{1}{8}. \quad (17)$$

Otherwise, replace ρ_0 by a smaller positive constant (leading to a smaller ε in part (a) of hypothesis 1.1). In hypothesis 1.1 (b) we picked $r \in (0, \rho_0)$. Hence

$$T_1 = T_1(r, \mu, \rho_0) := -\frac{2}{\mu} \ln \frac{r}{\rho_0} > 0 \quad (18)$$

is well-defined. This definition ensures the second of the two endpoint conditions (20); see step 2. Assume $\rho \in (0, \rho_0/2]$ is sufficiently small such that

$$c\kappa(\rho) \left(\frac{9}{\mu^{\frac{1}{4}}} + \frac{4}{3\mu}c + 4c \right) \leq \frac{1}{8} \quad (19)$$

and the closed ball of radius ρ about any point of the descending sphere S_ε^u is contained in $\mathcal{B}_{\rho_0} \subset \mathcal{U}$. Use compactness of the closure of the descending disk W_ε^u bounded by S_ε^u to fix a constant $T_2 = T_2(\rho, \varepsilon) \geq 0$ such that $\phi_{-T_2/4} W_\varepsilon^u \subset \mathcal{B}_\rho$; cf. (38) in step 5. Set $T_0 = T_0(r, \mu, \rho_0, \rho, \varepsilon) := \max\{T_1, T_2\} > 0$.

Pick $T \geq T_0$ and $\gamma \in S_\varepsilon^u$ and consider the infinite dimensional disk $\mathcal{D}_\gamma = \{\gamma\} \times \mathcal{B}_r^+$. The key observation to represent the preimage $\phi_T^{-1} \mathcal{D}_\gamma$ under the time- T -map ϕ_T as a graph over the stable subspace X^+ is the fact that to any pair $(q_u, z_+) \in X^- \oplus X^+$ sufficiently close to zero there corresponds a unique heatflow trajectory ξ whose initial value $\xi(0)$ projects under π_+ to z_+ and whose endpoint at time T projects under π_- to q_u ; see remark 1.6. In particular, for $q_u := \gamma$ any $z_+ \in X^+$ near the origin corresponds to a unique heatflow line $\xi = \xi_{\gamma, z_+}^T$ which ends at time T in \mathcal{D}_γ . Because its initial value $\xi(0)$ is of the form $(\pi_- \xi(0), z_+)$, it is natural to define the map $G_\gamma^T(z_+) := \pi_- \xi(0)$ whose graph at z_+ reproduces $\xi(0)$. In fact, we prove below that for each $z_+ \in X^+$ with $\|z_+\| \leq \rho/2c$ there is precisely one semiflow line $\xi = \xi_{\gamma, z_+}^T$ with initial condition $\pi_+ \xi(0) = z_+$ and endpoint condition $\xi(T) \in \mathcal{D}_\gamma$. The latter is equivalent to

$$\pi_- \xi(T) = \gamma \quad \wedge \quad \|\xi(T) - \gamma\|_X \leq r. \quad (20)$$

We will see in step 2 that the definition of T_1 assures the second condition.

The key step to determine the unique semiflow line ξ associated to the triple (T, γ, z_+) is to set up a strict contraction on a complete metric space Z^T whose (unique) fixed point is ξ . We define

$$\|\xi\|_{\text{exp}} = \|\xi\|_{\text{exp}, T} := \max_{s \in [0, T]} e^{s\frac{\mu}{2}} \|\xi(s)\|_X \quad (21)$$

and

$$Z^T = Z_{\gamma, \frac{\mu}{2}}^T := \left\{ \xi \in C^0([0, T], X) : \|\xi - \phi_\cdot \gamma\|_{\text{exp}} \leq \rho \right\} \quad (22)$$

³The smallness condition (17) on ρ_0 will only be used in the final step II (uniform convergence) of the proof of the L^2 extension theorem 1.3.

where $\phi_s \gamma_T = \phi_{s-T} \gamma$. Consider the map $\Psi^T = \Psi_{\gamma, z_+}^T$ defined on Z^T by

$$\begin{aligned} \left(\Psi_{\gamma, z_+}^T \xi \right) (s) &:= e^{-sA} z_+ + \int_0^s e^{-(s-\sigma)A} \pi_+ f(\xi(\sigma)) d\sigma \\ &+ e^{-(s-T)A^-} \gamma - \int_s^T e^{-(s-\sigma)A^-} \pi_- f(\xi(\sigma)) d\sigma \end{aligned} \quad (23)$$

for every $s \in [0, T]$. The fixed points of Ψ^T correspond to the desired heatflow trajectories by proposition 2.5. By step 1 and step 2 below Ψ^T is a strict contraction on Z^T . Hence by the Banach fixed point theorem it admits a unique fixed point ξ_{γ, z_+}^T and for $\mathcal{B}^+ := \mathcal{B}_{\rho/2c}^+ \subset X^+$ we define the map

$$G^T : S_\varepsilon^u \times \mathcal{B}^+ \rightarrow W_\delta^u \subset X^-, \quad (\gamma, z_+) \mapsto \pi_- \xi_{\gamma, z_+}^T(0) =: G_\gamma^T(z_+), \quad (24)$$

where \mathcal{B}^+ is the same ball for which the stable manifold theorem 2.6 holds true.

Remark 3.2. The case $p = 1$ covers the general case of remark 1.5 (i), because it represents the worst case in terms of the singularities in s which enter through proposition 2.4. Indeed $s^{-\frac{1+2p}{4p}} \leq s^{-\frac{3}{4}}$ for all $p \geq 1$ and $s \in (0, 1]$. The case $s > 1$ is irrelevant due to the presence of exponentially decaying factors. Therefore throughout the proof we fix $p = 1$ and set $X = W^{1,2}$ and $Y = L^1$.

The proof takes six steps. Fix $\gamma \in S_\varepsilon^u$ and $z_+ \in \mathcal{B}^+$.

Step 1. Assume $T \geq 0$. Then the set Z^T equipped with the metric induced by the exp norm is a complete metric space, any $\xi \in Z^T$ takes values in \mathcal{B}_{ρ_0} , and $\Psi^T = \Psi_{\gamma, z_+}^T$ acts on Z^T .

Proof. In case of the compact domain $[0, T]$ the space $C^0([0, T], X)$ is complete with respect to the supremum norm, hence with respect to the exp norm as both norms are equivalent by compactness of $[0, T]$. The subset $Z^T \subset C^0([0, T], X)$ is closed with respect to the exp norm. By the assumption which immediately follows (19) the elements of Z^T take values in \mathcal{B}_{ρ_0} , hence in \mathcal{U} .

To see that Ψ^T acts on Z^T we need to verify that $\Psi^T \xi$ is continuous and satisfies the exponential decay condition whenever $\xi \in Z^T$. By definition $\Psi^T \xi$ is a sum of four terms. That each one is continuous as a map $[0, T] \rightarrow X$ is standard. For terms one, two, and four see step 1 (iii) in the proof of theorem 2.6 given in [W]. For term three continuity follows from the definition of the exponential by the power series (12). For latter reference we sketch the argument for term two which we denote by $F(s)$: Continuity of $F : [0, T] \rightarrow X$ and the fact that $F(0) = 0$ (used in steps 2 and 3 below) both follow by an analogue of [W10a, le. 9.7 a)] for $-A$ instead of Δ and with $p = 2$; see also [He81, le. 3.2.1]. The condition to be checked is that the map $\tilde{f} := \pi_+ \circ f \circ \xi : [0, T] \rightarrow Y^+ \hookrightarrow Y$ is continuous and bounded: This is true since $\xi : [0, T] \rightarrow X$ is continuous and bounded by definition of Z^T and so is f by the Lipschitz lemma 2.1.

We prove exponential decay. For $s \in [0, T]$ consider the heatflow trajectory given by $\phi_s \gamma_T$. By the representation formula of proposition 2.5 it satisfies

$$\begin{aligned} \phi_s \gamma_T &= \int_0^s e^{-(s-\sigma)A} \pi_+ f(\phi_\sigma \gamma_T) d\sigma \\ &\quad + e^{-(s-T)A^-} \gamma - \int_s^T e^{-(s-\sigma)A^-} \pi_- f(\phi_\sigma \gamma_T) d\sigma. \end{aligned} \quad (25)$$

Here we used that $\pi_+ \gamma_T = 0$, because γ and therefore $\gamma_T = \phi_{-T} \gamma$ lies in $W^u(0, \mathcal{U}) \subset X^-$ by backward flow invariance. By the same argument $\pi_- \phi_T \gamma_T = \pi_- \gamma = \gamma$. By definition (23) of Ψ^T and (25) we get for $s \in [0, T]$ the estimate

$$\begin{aligned} &\|(\Psi^T \xi)(s) - \phi_s \gamma_T\|_X \\ &\leq \|e^{-sA} z_+\|_X + \int_0^s \|e^{-(s-\sigma)A} \pi_+\|_{\mathcal{L}(Y, X)} \|f(\xi(\sigma)) - f(\phi_\sigma \gamma_T)\|_Y d\sigma \\ &\quad + \int_s^T \|e^{-(s-\sigma)A^-} \pi_-\|_{\mathcal{L}(Y, X)} \|f(\xi(\sigma)) - f(\phi_\sigma \gamma_T)\|_Y d\sigma \\ &\leq c e^{-s\mu} \|z_+\|_X + c\kappa(\rho) e^{-s\frac{\mu}{2}} \|\xi - \phi \cdot \gamma_T\|_{\exp} \int_0^s \frac{e^{-(s-\sigma)\frac{\mu}{2}}}{(s-\sigma)^{\frac{3}{4}}} d\sigma \\ &\quad + c\kappa(\rho) e^{-s\frac{\mu}{2}} \|\xi - \phi \cdot \gamma_T\|_{\exp} \int_s^T e^{(s-\sigma)\frac{3}{2}\mu} d\sigma \\ &\leq \frac{\rho}{2} e^{-s\mu} + c\kappa(\rho) \left(\frac{8}{\mu^{\frac{1}{4}}} + \frac{2}{3\mu} \right) \rho e^{-s\frac{\mu}{2}} \leq \rho e^{-s\frac{\mu}{2}} \end{aligned} \quad (26)$$

where the last inequality is by smallness (19) of ρ . Inequality two follows by the exponential decay proposition 2.4 with constant c and the Lipschitz lemma 2.1 for f with Lipschitz constant $\kappa(\rho)$. We multiplied the integrands by $e^{-\sigma\frac{\mu}{2}} e^{\sigma\frac{\mu}{2}}$ to create the exp norms. Inequality three uses $\|z_+\|_X \leq \frac{\rho}{2c}$ and boundedness of the exp norms by ρ since $\xi \in Z^T$. We also used the identity

$$\int_s^\infty e^{(s-\sigma)\frac{3}{2}\mu} d\sigma = \frac{2}{3\mu}. \quad (27)$$

To estimate the other integral define $\Gamma(\alpha) := \int_0^\infty e^{-\tau} \tau^{\alpha-1} d\tau$ for $\alpha > 0$. The Γ function satisfies $\Gamma(\frac{1}{4}) = 4\Gamma(\frac{5}{4}) \leq 4$. Hence

$$\int_0^s \frac{e^{-(s-\sigma)\frac{\mu}{2}}}{(s-\sigma)^{\frac{3}{4}}} d\sigma = \left(\frac{2}{\mu} \right)^{\frac{1}{4}} \int_0^{s\frac{\mu}{2}} e^{-\tau} \tau^{\frac{1}{4}-1} d\tau \leq \frac{2}{\mu^{\frac{1}{4}}} \Gamma(\frac{1}{4}) \leq \frac{8}{\mu^{\frac{1}{4}}}. \quad (28)$$

□

Step 2. For $T \geq 0$ the map $\Psi^T = \Psi_{\gamma, z_+}^T$ acting on Z^T is a strict contraction. Each image point $\Psi^T \xi$ satisfies the initial condition $\pi_+(\Psi^T \xi)(0) = z_+$ and for $T \geq T_1$ also the endpoint conditions (20), i.e. it hits $\mathcal{D} = S_\varepsilon^u \times B_r^+$ at time T .

Proof. Assume $T \geq 0$ and fix $\xi_1, \xi_2 \in Z^T$. Similarly to (26) we obtain that

$$\begin{aligned}
& \|(\Psi^T \xi_1)(s) - (\Psi^T \xi_2)(s)\|_X \\
& \leq \int_0^s \left\| e^{-(s-\sigma)A} \pi_+ \right\|_{\mathcal{L}(Y, X)} \|f(\xi_1(\sigma)) - f(\xi_2(\sigma))\|_Y d\sigma \\
& \quad + \int_s^T \left\| e^{-(s-\sigma)A^-} \pi_- \right\|_{\mathcal{L}(Y, X)} \|f(\xi_1(\sigma)) - f(\xi_2(\sigma))\|_Y d\sigma \\
& \leq c\kappa(\rho) \left(\frac{8}{\mu^{\frac{1}{4}}} + \frac{2}{3\mu} \right) e^{-s\frac{\mu}{2}} \|\xi_1 - \xi_2\|_{\text{exp}}
\end{aligned} \tag{29}$$

for $s \in [0, T]$. Now use the smallness assumption (19) on ρ to conclude that $\|\Psi^T \xi_1 - \Psi^T \xi_2\|_{\text{exp}} \leq \frac{1}{2} \|\xi_1 - \xi_2\|_{\text{exp}}$.

The identities $\pi_+(\Psi^T \xi)(0) = z_+$ and $\pi_-(\Psi^T \xi)(T) = \gamma$ follow from definition (23) of Ψ^T , the identities $\pi_+ \pi_- = \pi_- \pi_+ = 0$, strong continuity of the semigroups on X^- and X^+ by proposition 2.4, continuity and boundedness of both integrands, and $F(0) = 0$ by the proof of step 1. Concerning the second endpoint condition in (20) assume $T \geq T_1$ and evaluate (26) at $s = T$ to get

$$\|(\Psi \xi)(T) - \gamma\|_X \leq e^{-T\frac{\mu}{2}} \rho \leq e^{-T_1\frac{\mu}{2}} \rho_0 = r$$

where the last step is by definition of T_1 in (18). \square

Step 3. For $T \geq 0$ the map $G^T : S_\varepsilon^u \times \mathcal{B}^+ \rightarrow X^-$ defined by (24) is of class C^1 and, for each $\gamma \in S_\varepsilon^u$, the map $G_\gamma^T := G^T(\gamma, \cdot) : \mathcal{B}^+ \rightarrow X^-$ satisfies

$$G_\gamma^T(0) = \phi_{-T}\gamma =: \gamma_T, \quad \text{graph } G_\gamma^T = \left\{ \xi_{\gamma, z_+}^T(0) \mid z_+ \in \mathcal{B}^+ \right\}.$$

Proof. Assume $T \geq 0$. By step 2 and its proof the map

$$\Psi^T : S_\varepsilon^u \times \mathcal{B}^+ \times Z^T \rightarrow Z^T, \quad (\gamma, z_+, \xi) \mapsto \Psi_{\gamma, z_+}^T \xi$$

is a uniform contraction on Z^T with contraction factor $\frac{1}{2}$. (Actually Z^T depends on γ , but the complete metric spaces associated to different γ 's are naturally isomorphic.) Observe that Ψ^T is linear, hence smooth, in γ and in z_+ and of class C^1 in ξ , because f is of class C^1 by the Lipschitz lemma 2.1. Hence by the uniform contraction principle, see e.g. [CH82], the map $\lambda : S_\varepsilon^u \times \mathcal{B}^+ \rightarrow Z^T$ which assigns to (γ, z_+) the unique fixed point ξ_{γ, z_+}^T of Ψ_{γ, z_+}^T is of class C^1 and so is its composition with the (linear) evaluation map $ev_0 : Z^T \rightarrow X$, $\xi \mapsto \xi(0)$, and the (linear) projection $\pi_- : X \rightarrow X^-$. But this composition is G^T by definition (24). This proves that G^T , thus \mathcal{G} , is of class C^1 in γ and z_+ .

Consider the heatflow trajectory $\eta^u : [0, T] \rightarrow X$, $s \mapsto \phi_s \gamma_T = \phi_{s-T} \gamma$. It takes values in $W_\varepsilon^u \subset X^-$, because $\gamma \in S_\varepsilon^u = \partial W_\varepsilon^u$ and the descending disk W_ε^u is backward flow invariant. Hence $\pi_+ \eta^u(0) = 0$ and π_- leaves η^u pointwise invariant. Thus $\eta^u = \xi_{\gamma, 0}^T$ by uniqueness of the fixed point and therefore

$$G_\gamma^T(0) := \pi_- \xi_{\gamma, 0}^T(0) = \pi_- \eta^u(0) = \eta^u(0) = \gamma_T.$$

To get the desired representation of graph G_γ^T observe that

$$\mathcal{G}_\gamma^T(z_+) := (G_\gamma^T(z_+), z_+) = \left(\pi_- \xi_{\gamma, z_+}^T(0), \pi_+ \xi_{\gamma, z_+}^T(0) \right) = \xi_{\gamma, z_+}^T(0) \quad (30)$$

by definition (24). The first identity also uses the fixed point property and the initial condition proved in step 2. The final identity is by $\pi_- \oplus \pi_+ = \mathbb{1}_Y$. \square

Step 4. *The map \mathcal{G} is of class C^1 . The map $T \mapsto \mathcal{G}(T, \gamma, z_+)$ is Lipschitz continuous and its derivative is locally Hölder continuous with exponent $\alpha = \frac{1}{8}$.*

Proof. By linearity the map $\gamma \mapsto \mathcal{G}(T, \gamma, z_+)$ is smooth, hence Lipschitz continuous by compactness of S_ε^u . By compactness of $[0, T]$ and by step 3 it remains to prove that \mathcal{G} is of class C^1 in the T variable. Pick $T \geq T_0 > 0$. The unique fixed point $\xi^T := \xi_{\gamma, z_+}^T$ of Ψ^T is given by (23) and the one of $\Psi^{T+\tau}$ by

$$\begin{aligned} \xi^{T+\tau}(s) := \xi_{\gamma, z_+}^{T+\tau}(s) &= e^{-sA} z_+ + \int_0^s e^{-(s-\sigma)A} \pi_+ f(\xi^{T+\tau}(\sigma)) d\sigma \\ &\quad + e^{-(s-T-\tau)A^-} \gamma - \int_s^{T+\tau} e^{-(s-\sigma)A^-} \pi_- f(\xi^{T+\tau}(\sigma)) d\sigma. \end{aligned}$$

For $s \in [0, T]$ and $\tau \geq 0$ we obtain, analogously to (26), the estimate

$$\begin{aligned} &\|\xi^{T+\tau}(s) - \xi^T(s)\|_X \\ &\leq \int_0^s \|e^{-(s-\sigma)A} \pi_+\|_{\mathcal{L}(Y, X)} \|f(\xi^{T+\tau}(\sigma)) - f(\xi^T(\sigma))\|_Y d\sigma \\ &\quad + \|(e^{\tau A^-} - \mathbb{1})e^{-(s-T)A^-} \gamma\|_X \\ &\quad + \int_s^T \|e^{-(s-\sigma)A^-} \pi_-\|_{\mathcal{L}(Y, X)} \|f(\xi^{T+\tau}(\sigma)) - f(\xi^T(\sigma))\|_Y d\sigma \\ &\quad + \int_T^{T+\tau} \|e^{-(s-\sigma)A^-} \pi_-\|_{\mathcal{L}(Y, X)} \|f(\xi^{T+\tau}(\sigma))\|_Y d\sigma \\ &\leq c\kappa(\rho) \|\xi^{T+\tau} - \xi^T\|_{C^0([0, T], X)} \left(\int_0^s \frac{e^{-(s-\sigma)\mu}}{(s-\sigma)^{\frac{3}{4}}} d\sigma + \int_s^T e^{(s-\sigma)\mu} d\sigma \right) \\ &\quad + \tau c|\lambda_1| \cdot ce^{(s-T)\mu} \|\gamma\|_X + c\kappa(\rho)\rho_0 \int_T^{T+\tau} e^{(s-\sigma)\mu} d\sigma \\ &\leq c\kappa(\rho) \left(\frac{8}{\mu^{\frac{1}{4}}} + \frac{1}{\mu} \right) \|\xi^{T+\tau} - \xi^T\|_{C^0([0, T], X)} + \tau\rho_0 c^2 |\lambda_1| e^{(s-T)\mu} \\ &\quad + c\kappa(\rho)\rho_0 \frac{e^{(s-T)\mu}}{\mu} (1 - e^{-\tau\mu}) \\ &\leq \frac{1}{8} \|\xi^{T+\tau} - \xi^T\|_{C^0([0, T], X)} + \tau\rho_0 (c^2 |\lambda_1| + 1) e^{(s-T)\mu}. \end{aligned}$$

Inequality two uses the Lipschitz lemma 2.1 for f and the exponential estimates of proposition 2.4. To estimate the second of the four terms recall that X^- is

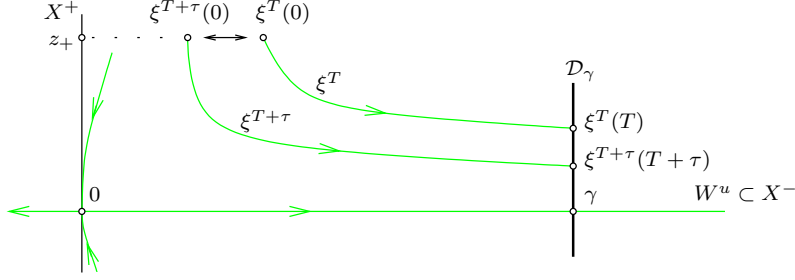


Figure 5: The difference $\mathcal{G}_\gamma^{T+\tau}(z_+) - \mathcal{G}_\gamma^T(z_+) = \xi^{T+\tau}(0) - \xi^T(0)$

spanned by an orthonormal basis of eigenvectors of $A^- \in \mathcal{L}(X^-)$ corresponding to the eigenvalues $\lambda_1 \leq \dots \leq \lambda_k < 0$. Hence $\|A^-\| = -\lambda_1 = |\lambda_1|$. By the mean value theorem

$$1 - e^{-\mu\tau} =: h(\tau) = h(0) + h'(\alpha_\tau\tau)\tau = \mu e^{-\mu\alpha_\tau\tau} \leq \mu\tau \quad (31)$$

for some α_τ between 0 and τ . It follows that

$$\|e^{\tau A^-} - \mathbb{1}\| = \left\| A^- \int_0^\tau e^{\sigma A^-} d\sigma \right\| \leq |\lambda_1| \int_0^\tau c e^{-\sigma\mu} d\sigma \leq \tau c |\lambda_1|. \quad (32)$$

The first identity even without norms is standard; see e.g. [L05, prop. 1.3.6. (ii)]. In the last step we calculated the integral using (31). Coming back to inequality two above, we used that $\xi^{T+\tau} \in Z^{T+\tau}$ in term four takes values in \mathcal{B}_{ρ_0} by step 1. Inequality uses (28) for the first integral. Inequality four uses (31) and twice smallness (19) of ρ . Now take the supremum over $s \in [0, T]$ to get the estimate

$$\|\xi^{T+\tau} - \xi^T\|_{C^0([0, T], X)} \leq \rho_0 c_1 \tau \quad (33)$$

with constant $c_1 = 2(c^2|\lambda_1| + 1)$. By (30) we obtain the estimate

$$\|\mathcal{G}_\gamma^{T+\tau}(z_+) - \mathcal{G}_\gamma^T(z_+)\|_X = \|\xi^{T+\tau}(0) - \xi^T(0)\|_X \leq \rho_0 c_1 \tau$$

and this proves that $\mathcal{G}(T, \gamma, z_+) = \mathcal{G}_\gamma^T(z_+)$ is Lipschitz continuous in T . The difference $\xi^{T+\tau} - \xi^T$ is illustrated by figure 5.

To see that \mathcal{G} is continuously differentiable in T consider the derivative

$$\begin{aligned} \Theta^T(s) &:= \frac{d}{d\tau} \Big|_{\tau=0} \xi_{\gamma, z_+}^{T+\tau}(s) \\ &= \int_0^s e^{-(s-\sigma)A} \pi_+ (df|_{\xi^T(\sigma)} \circ \Theta^T(\sigma)) d\sigma + A^- e^{-(s-T)A^-} \gamma \\ &\quad - e^{-(s-T)A^-} \pi_- f(\xi^T(T)) - \int_s^T e^{-(s-\sigma)A^-} \pi_- (df|_{\xi^T(\sigma)} \circ \Theta^T(\sigma)) d\sigma. \end{aligned}$$

Since $\frac{d}{dT}\mathcal{G}(T, \gamma, z_+) = \Theta^T(0)$ by (30), it remains to show that the map $T \mapsto \Theta^T(0) \in X$ is locally Hölder continuous. By definition of Θ^T we get the identity

$$\begin{aligned}
\Theta^{T+\tau}(s) - \Theta^T(s) &= \int_0^s e^{-(s-\sigma)A} \pi_+ df|_{\xi^{T+\tau}(\sigma)} (\Theta^{T+\tau}(\sigma) - \Theta^T(\sigma)) d\sigma \\
&\quad + \int_0^s e^{-(s-\sigma)A} \pi_+ (df|_{\xi^{T+\tau}(\sigma)} - df|_{\xi^T(\sigma)}) \circ \Theta^T(\sigma) d\sigma \\
&\quad + (e^{\tau A^-} - \mathbb{1}) A^- e^{-(s-T)A^-} \gamma \\
&\quad - (e^{\tau A^-} - \mathbb{1}) e^{-(s-T)A^-} \pi_- f(\xi^{T+\tau}(T+\tau)) \\
&\quad - e^{-(s-T)A^-} \pi_- (f(\xi^{T+\tau}(T+\tau)) - f(\xi^T(T))) \\
&\quad - \int_s^T e^{-(s-\sigma)A^-} \pi_- df|_{\xi^{T+\tau}(\sigma)} (\Theta^{T+\tau}(\sigma) - \Theta^T(\sigma)) d\sigma \\
&\quad - \int_s^T e^{-(s-\sigma)A^-} \pi_- (df|_{\xi^{T+\tau}(\sigma)} - df|_{\xi^T(\sigma)}) \circ \Theta^T(\sigma) d\sigma \\
&\quad - \int_T^{T+\tau} e^{-(s-\sigma)A^-} \pi_- df|_{\xi^{T+\tau}(\sigma)} \circ \Theta^{T+\tau}(\sigma) d\sigma
\end{aligned}$$

for all $s \in [0, T]$ and $\tau \geq 0$. To obtain lines one and two we added zero, similarly for lines four and five and lines six and seven. Abbreviate the $C^0([0, T], X)$ norm by $\|\cdot\|_{C_T^0}$. Combine lines one and six and lines two and seven to get that

$$\begin{aligned}
&\|\Theta^{T+\tau}(s) - \Theta^T(s)\|_X \\
&\leq c\kappa(\rho) \|\Theta^{T+\tau} - \Theta^T\|_{C_T^0} \left(\int_0^s \frac{e^{-(s-\sigma)\mu}}{(s-\sigma)^{\frac{3}{4}}} d\sigma + \int_s^T e^{(s-\sigma)\mu} d\sigma \right) \\
&\quad + c\kappa_* \|\xi^{T+\tau} - \xi^T\|_{C_T^0} \|\Theta^T\|_{C_T^0} \left(\int_0^s \frac{e^{-(s-\sigma)\mu}}{(s-\sigma)^{\frac{3}{4}}} d\sigma + \int_s^T e^{(s-\sigma)\mu} d\sigma \right) \\
&\quad + \tau c^2 |\lambda_1| (|\lambda_1| \cdot \|\gamma\|_X + \kappa(\rho) \|\xi^{T+\tau}(T+\tau)\|_X) e^{(s-T)\mu} \\
&\quad + c\kappa(\rho) \|\xi^{T+\tau}(T+\tau) - \xi^T(T)\|_X e^{(s-T)\mu} \\
&\quad + c\kappa(\rho) \|\Theta^{T+\tau}\|_{C_{T+\tau}^0} \int_T^{T+\tau} e^{(s-\sigma)\mu} d\sigma \\
&\leq c\kappa(\rho) \left(\frac{8}{\mu^{\frac{1}{4}}} + \frac{1}{\mu} \right) \|\Theta^{T+\tau} - \Theta^T\|_{C_T^0} + \tau c\kappa_* \rho_0^2 c_1^2 \left(\frac{8}{\mu^{\frac{1}{4}}} + \frac{1}{\mu} \right) \\
&\quad + \tau c\rho_0 |\lambda_1| (c|\lambda_1| + 1) + \tau^{\frac{1}{8}} \rho_0 \left(c_3 T^{\frac{3}{8}} + c_4 \tau^{\frac{1}{8}} \right) + \frac{c_1}{\mu} (1 - e^{-\tau\mu}) \\
&\leq \frac{1}{8} \|\Theta^{T+\tau} - \Theta^T\|_{C_T^0} + \tau \rho_0 c_1^2 + \tau c\rho_0 |\lambda_1| (c|\lambda_1| + 1) \\
&\quad + \tau^{\frac{1}{8}} \rho_0 \left(c_3 T^{\frac{3}{8}} + c_4 \tau^{\frac{1}{8}} \right) + \tau \rho_0 c_1
\end{aligned}$$

for $s \in [0, T]$ and $\tau \geq 0$. Inequality one uses the exponential decay proposition 2.4, the Lipschitz lemma 2.1 for f , and its corollary 2.2. To obtain line

three we used (32). To obtain line four we added zero. To see inequality two observe the following. Estimate the first integral in lines one and two by (28). Apply estimate (33) itself and use it to conclude that $\|\Theta^T(s)\|_X \leq \rho_0 c_1$ whenever $s \in [0, T]$. (Note that the same is true when T is replaced by $T + \tau$.) Recall that $\gamma \in \mathcal{B}_{\rho_0}$ by our local setup and that the elements of Z^T (and $Z^{T+\tau}$) take values in \mathcal{B}_{ρ_0} by step 1. Note that $c\kappa(\rho) \leq 1$ by (19) and that $e^{(s-T)\mu} \leq 1$. To estimate the difference $\xi^{T+\tau}(T + \tau) - \xi^T(T) \in X$ in line four is surprisingly subtle. This estimate will be carried out separately below; see (35) for the result used in inequality two above. To obtain inequality three we used smallness (17) and (19) of ρ and estimate (31). Now take the supremum over $s \in [0, T]$ to get

$$\|\Theta^{T+\tau} - \Theta^T\|_{C^0([0, T], X)} \leq c_T \rho_0 \tau^{\frac{1}{8}} \quad (34)$$

where $c_T = c_3 T^{\frac{3}{8}} + \tau^{\frac{7}{8}} (c_1^2 + c^2 \lambda_1^2 + c|\lambda_1| + c_1) + c_4 \tau^{\frac{1}{8}}$; see (35). Thus

$$\left\| \frac{d}{dT} \mathcal{G}(T + \tau, \gamma, z_+) - \frac{d}{dT} \mathcal{G}(T, \gamma, z_+) \right\|_X = \|\Theta^{T+\tau}(0) - \Theta^T(0)\|_X \leq c_T \rho_0 \tau^{\frac{1}{8}},$$

that is $\frac{d}{dT} \mathcal{G}(T, \gamma, z_+)$ is locally Hölder continuous with exponent $\alpha = \frac{1}{8}$.

As mentioned above it remains to estimate the $W^{1,2}$ norm of the difference

$$\begin{aligned} \xi^{T+\tau}(T + \tau) - \xi^T(T) &= (e^{-\tau A} - \mathbb{1}) e^{-TA} z_+ \\ &\quad + \int_0^T e^{-(T+\tau-\sigma)A} \pi_+ (f(\xi^{T+\tau}(\sigma)) - f(\xi^T(\sigma))) d\sigma \\ &\quad + \int_0^T (e^{-\tau A} - \mathbb{1}) e^{-(T-\sigma)A} \pi_+ f(\xi^T(\sigma)) d\sigma \\ &\quad + \int_T^{T+\tau} e^{-(T+\tau-\sigma)A} \pi_+ f(\xi^{T+\tau}(\sigma)) d\sigma. \end{aligned}$$

We added zero to obtain terms II and III in this sum I+II+III+IV of four.

I) Concerning term one we get

$$\begin{aligned} \|(e^{-\tau A} - \mathbb{1}) e^{-TA} z_+\|_X &= \left\| \int_0^\tau -A e^{-sA} e^{-TA} z_+ ds \right\|_X \\ &\leq \int_0^\tau \|e^{-sA} \pi_+\|_{\mathcal{L}(X)} \|A e^{-TA} \pi_+\|_{\mathcal{L}(X)} \|z_+\|_X ds \\ &\leq \int_0^\tau c e^{-s\mu} \left(\frac{c' C}{T} e^{-T\mu} \right) \rho_0 ds \\ &\leq \tau \frac{cc' C \rho_0 e^{-T_0 \mu}}{T_0}. \end{aligned}$$

The first identity even without norms is standard; see e.g. [L05, prop. 1.3.6. (ii)]. To obtain inequality one we permuted A and e^{-sA} ; see e.g. [L05, thm. 1.3.3. (i)]. Here we used that $e^{-TA} z_+ \in W^{2,1} = D(A)$ since $T > 0$. Compare the above estimate on X^+ with the corresponding estimate (32) on the finite dimensional vector space X^- and note how boundedness of A^- simplifies (32). Inequality

two uses that the norms $\|A\cdot\|_{1,2}$ and $\|\cdot\|_{3,2}$ are equivalent with constant c' by compactness of S^1 and A being of second order. The regularity-for-singularity estimate (13) with constant $C = C(\mu)$ then allows to get from $W^{3,2}$ back to $W^{1,2}$ catching the factor CT^{-1} . The final inequality uses (31).

II) By (33) and the fact that $c\kappa(\rho) \leq 1$ by (19) we get for term two

$$\begin{aligned} \int_0^T \left\| e^{-(T+\tau-\sigma)A} \pi_+ (f(\xi^{T+\tau}(\sigma)) - f(\xi^T(\sigma))) \right\|_X d\sigma &\leq \tau \rho_0 c_1 \int_\tau^{(T+\tau)} \frac{e^{-s\mu}}{s^{\frac{3}{4}}} ds \\ &\leq \tau^{\frac{1}{4}} \rho_0 c_1 e^{-\tau\mu} T. \end{aligned}$$

Here we estimated $s^{-\frac{3}{4}}$ by $(\tau\mu)^{-\frac{3}{4}}$. After integration one encounters a factor $(1 - e^{-T\mu})/\mu$ which by (31) is bounded above by T . Two side remarks are in order. Firstly, not using (31) we could trade T for μ^{-1} . Secondly, one could trade $\tau^{\frac{1}{4}}$ for τ paying with an extra factor T^α for some $\alpha \in (0, 1]$. However, term three shows that these alternatives are pointless.

III) Term three requires similar techniques as term one, but their application requires more care. Namely, it is crucial to “distribute” the second order operator A over the two semigroups with parameters s and T , respectively, in order to avoid catching either a factor $s^{-\alpha}$ or a factor $T^{-\alpha}$ with $\alpha \geq 1$ when trading regularity for singularity via (13), because each of these factors is not integrable over an interval containing zero. Similarly as in I) we get

$$\begin{aligned} &\int_0^T \left\| (e^{-\tau A} - \mathbb{1}) e^{-(T-\sigma)A} \pi_+ f(\xi^T(\sigma)) \right\|_X d\sigma \\ &\leq \int_0^T \int_0^\tau \left\| A e^{-sA} e^{-(T-\sigma)A} \pi_+ f(\xi^T(\sigma)) \right\|_{W^{1,2}} ds d\sigma \\ &\leq \int_0^T \int_0^\tau c' \left\| e^{-sA} e^{-(T-\sigma)A} \pi_+ f(\xi^T(\sigma)) \right\|_{W^{3,2}} ds d\sigma \\ &\leq c' \kappa(\rho) \rho_0 \int_0^T \int_0^\tau \left\| e^{-sA} \pi_+ \right\|_{\mathcal{L}(W^{1,q}, W^{3,2})} \left\| e^{-(T-\sigma)A} \pi_+ \right\|_{\mathcal{L}(L^1, W^{1,q})} ds d\sigma \\ &\leq c' C' C'' \rho_0 \int_0^\tau e^{-s\mu} s^{-\frac{3}{4} - \frac{1}{2q}} ds \int_0^T e^{-(T-\sigma)\mu} (T-\sigma)^{-\frac{1}{2} - \frac{1}{2q}} d\sigma \\ &= c' C' C'' \rho_0 \int_0^\tau e^{-s\mu} s^{-\frac{7}{8}} ds \int_0^T e^{-s\mu} s^{-\frac{5}{8}} ds \\ &\leq \tau^{\frac{1}{8}} 22 c' C' C'' \rho_0 T^{\frac{3}{8}} \end{aligned}$$

where in inequality three we used once more that $\|\xi^T(\sigma)\|_X \leq \rho_0$ by step 1. Pick $q > 2$, say $q = 4$, and note that $\kappa(\rho) \leq 1$. In inequality four we applied twice the regularity-for-singularity estimate (13) with constants C' and C'' , respectively. Finally we dropped the factors $e^{-s\mu} \leq 1$ and carried out the integrals.

IV) Concerning term four we drop $e^{-s\mu} \leq 1$ in the last step to get

$$\int_T^{T+\tau} \left\| e^{-(T+\tau-\sigma)A} \pi_+ f(\xi^{T+\tau}(\sigma)) \right\|_X d\sigma \leq \rho_0 \int_0^\tau e^{-s\mu} s^{-\frac{3}{4}} ds \leq \tau^{\frac{1}{4}} 4 \rho_0.$$

Here is a side remark concerning the estimate for term three: Unfortunately, we do not see any way to trade $\tau^{\frac{1}{8}}$ for τ or, equivalently, to trade $T^{\frac{3}{8}}$ for T . This has the following consequences. The positive power of T obstructs the conclusion that $\frac{d}{dT}\mathcal{G}$ is uniformly continuous in T . The conclusion of local Lipschitz continuity is obstructed by the factor τ^α with $\alpha = \frac{1}{8} < 1$. All we can say is that $\frac{d}{dT}\mathcal{G}$ is locally Hölder continuous in T with exponent $\alpha = \frac{1}{8}$.

To summarize, the above estimates show that

$$\|\xi^{T+\tau}(T+\tau) - \xi^T(T)\|_X \leq \tau^{\frac{1}{8}} \rho_0 \left(c_3 T^{\frac{3}{8}} + c_4 \tau^{\frac{1}{8}} \right) \quad (35)$$

for $\tau \geq 0$ and where $c_3 := 22c'C'C'' + \tau^{\frac{1}{4}}c_1T^{\frac{5}{8}}$ and $c_4 := 4 + \tau^{\frac{7}{8}}cc'CT_0^{-1}$. This concludes the proof of (34) and therefore of step 4. \square

Step 5. For $T \geq 0$ the graph map $\mathcal{G}_\gamma^T : \mathcal{B}^+ \rightarrow X^- \oplus X^+$, $z_+ \mapsto (G_\gamma^T(z_+), z_+)$, and its inverse $\pi_+|_{\mathcal{G}_\gamma^T(\mathcal{B}^+)}$ are both Lipschitz continuous with respect to the $W^{1,2}$ norm. In fact the graph map is a diffeomorphism onto its image.

Proof. For $j = 1, 2$ pick $z_j \in \mathcal{B}^+$ and denote the fixed point ξ_{γ, z_j}^T of $\Psi^T = \Psi_{\gamma, z_j}^T$ by ξ_j . Similarly to the estimate in the proof of step 2 we obtain for $s \in [0, T]$

$$\|\xi_1(s) - \xi_2(s)\|_X \leq ce^{-s\mu} \|z_1 - z_2\|_X + \frac{1}{2} \|\xi_1 - \xi_2\|_{\text{exp}}$$

Multiplication by $e^{s\frac{\mu}{2}}$ and taking the supremum over $s \in [0, T]$ then shows that

$$\|\xi_1 - \xi_2\|_{\text{exp}} \leq 2c \|z_1 - z_2\|_X. \quad (36)$$

By (30) this proves Lipschitz continuity of \mathcal{G}_γ^T , namely

$$\|\mathcal{G}_\gamma^T(z_1) - \mathcal{G}_\gamma^T(z_2)\|_X = \|\xi_1(0) - \xi_2(0)\|_X \leq \|\xi_1 - \xi_2\|_{\text{exp}} \leq 2c \|z_1 - z_2\|_X.$$

Next use that π_+ vanishes on X^- and acts as the identity on X^+ to see that π_+ is a left inverse of \mathcal{G}_γ^T . Thus π_+ restricted to $\mathcal{G}_\gamma^T(\mathcal{B}^+)$ is its inverse. But this restriction is of class C^1 , because it is of the form $\pi_+ \circ \mathcal{G}_\gamma^T(z_+)$ where π_+ is linear and the map $z_+ \mapsto \mathcal{G}_\gamma^T(z_+) := (G_\gamma^T z_+, z_+)$ is of class C^1 by step 3.

To see that the restriction of π_+ to $\mathcal{G}_\gamma^T(\mathcal{B}^+)$ is Lipschitz continuous consider the difference $\xi_1(0) - \xi_2(0) = \Psi^T \xi_1(0) - \Psi^T \xi_2(0)$ whose right hand side is given by (23). Apply $\|a - b\| \geq \|a\| - \|b\|$ with $a = z_1 - z_2$ and (29) for $s = 0$ to get

$$\|\xi_1(0) - \xi_2(0)\|_X \geq \|z_1 - z_2\|_X - c\kappa(\rho) \frac{2}{3\mu} \|\xi_1 - \xi_2\|_{\text{exp}}.$$

By (36) and the smallness assumption (19) on ρ this implies that

$$\|\xi_1(0) - \xi_2(0)\|_X \geq \left(1 - c\kappa(\rho) \frac{2}{3\mu} 2c \right) \|z_1 - z_2\|_X \geq \frac{1}{2} \|z_1 - z_2\|_X$$

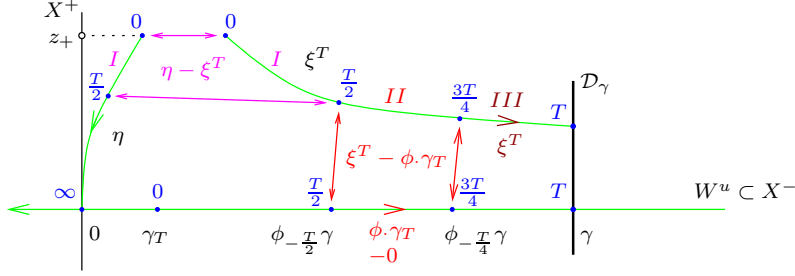


Figure 6: Time partitions and exponentially decaying differences

which by (30) and the fact that π_+ left inverts \mathcal{G}_γ^T is equivalent to

$$\|\mathcal{G}_\gamma^T(z_1) - \mathcal{G}_\gamma^T(z_2)\|_X \geq \frac{1}{2} \|\pi_+ \mathcal{G}_\gamma^T(z_1) - \pi_+ \mathcal{G}_\gamma^T(z_2)\|_X. \quad (37)$$

This proves that π_+ is Lipschitz continuous on the image of \mathcal{G}_γ^T .

By (37) and the estimate after (36) the map \mathcal{G}_γ^T is bi-Lipschitz, hence a homeomorphism onto its image. Since the map and its inverse are both of class C^1 , it is in fact a diffeomorphism onto its image. \square

Step 6. (Uniform convergence) $\|\mathcal{G}^\infty(z_+) - \mathcal{G}_\gamma^T(z_+)\|_X \leq 3c\rho_0 e^{-T\frac{\mu}{4}} \forall T \geq T_2$.

Proof. Assume $T \geq T_2$; see (38) below. Consider the fixed point $\xi^T = \xi_{\gamma, z_+}^T$ of Ψ_{γ, z_+}^T on Z^T and the fixed point $\eta = \eta_{z_+}$ of Ψ_{z_+} on Z defined by (16). Because $\mathcal{G}_\gamma^T(z_+) = \xi^T(0)$ by (30), similarly $\mathcal{G}^\infty(z_+) = \eta(0)$, it remains to estimate the difference $\eta(0) - \xi^T(0)$. As the respective representation formulae show this difference depends on the whole trajectories η and ξ^T . But while η runs into the origin, the trajectory ξ^T ends on the fiber \mathcal{D}_γ far away! So the difference $\eta - \xi^T$ cannot converge to zero, as $T \rightarrow \infty$, uniformly on $[0, T]$. However, as figure 5 suggests that this could be true on some initial part of the domain $[0, T]$, say on $[0, \frac{1}{2}T]$. Hence the first step is to reduce the problem to the smaller interval $[0, \frac{1}{2}T]$. The second step is to solve the reduced problem. Here the key idea is to *suitably partition* both trajectories η and ξ^T and compare due parts; see figure 6. The fact that η is *asymptotically well behaved*, i.e. exponentially close to zero on $[T, \infty)$, enters frequently.

We proceed as follows: In step (a) we estimate $\|\eta(0) - \xi^T(0)\|_X$ by an exponentially decaying function of T plus the supremum of $\|\eta(s) - \xi^T(s)\|_X$ over $s \in [0, \frac{1}{2}T]$. In step (b) we prove exponential decay of this sup norm. But here we encounter again the difference $\eta - \xi^T$, but unfortunately on the whole interval $[0, T]$. The *key idea* is to decompose this interval into three pieces, namely

$$I := [0, \frac{1}{2}T], \quad II := [\frac{1}{2}T, \frac{3}{4}T], \quad III := [\frac{3}{4}T, T],$$

as shown in figure 6. In fact an extra piece $[T, \infty)$ is brought in by η . On interval I we pull out the supremum norm and use smallness of the Lipschitz

constant $\kappa(\rho)$ to get a coefficient less than one to throw the whole $(\eta - \xi^T)$ term on the left hand side. Off I we apply the triangle inequality to deal with each term η and ξ^T separately. Exponential decay built into the definition (16) of Z allows to handle η on its whole remaining time interval $[\frac{1}{2}T, \infty)$ in one go. It remains to deal with ξ^T on intervals II and III . For $\sigma \in II$ we exploit (after adding zero) that both $\xi^T(\sigma) - \phi_\sigma(\gamma_T)$ and $\phi_\sigma(\gamma_T)$ individually decay exponentially in T , uniformly in $\sigma \in II$. For the first term this is simply true by definition (22) of Z^T . Concerning the second term we use that γ lies in the unstable manifold. Hence $\phi_\sigma(\gamma_T) = \phi_t(\gamma)$ collapses exponentially fast into the origin, since $t := \sigma - T \in [-\frac{1}{2}T, -\frac{1}{4}T]$ and the whole interval sets off to $-\infty$ ⁴; cf. remark 2.8. For interval III the argument is *analytic* and *cannot be guessed* by figure 6. The figure even suggests trouble. Fortunately, we are not concerned with the image of the trajectory, but with the integral over its time parametrization. In fact due to an abundance of negative powers we even get away with the coarse estimate $\|\xi^T(\sigma)\|_X \leq \rho_0$. This leaves us with integrating $e^{(s-\sigma)\mu}$ over III . But $s \leq \frac{1}{2}T$ by assumption and $\sigma \geq \frac{3}{4}T$ on III .⁵

Our choice of time partitions and combinations of trajectory pieces which leads to exponential decay in T is shown in figure 6. The upper (blue) labels are time. It is instructive to figure out how the drawing changes as T tends to infinity. How do η and ξ^T change and how their time labels? What happens to the length of the four double arrows? Consider the pair of double arrows with common point $\xi^T(T/2)$. What is the asymptotic behavior of this point?

(a) By formula (23) for ξ^T and the one for η , see formula after (16), we get

$$\begin{aligned}
& \|\eta(0) - \xi^T(0)\|_X \\
& \leq \left(\int_0^{T/2} + \int_{T/2}^T \right) \|e^{\sigma A^-} \pi_- \|_{\mathcal{L}(Y,X)} \|f \circ \eta(\sigma) - f \circ \xi^T(\sigma)\|_Y d\sigma \\
& \quad + \|e^{TA^-} \pi_- \|_{\mathcal{L}(X)} \|\gamma\|_X + \int_T^\infty \|e^{\sigma A^-} \pi_- \|_{\mathcal{L}(Y,X)} \|f \circ \eta(\sigma)\|_Y d\sigma \\
& \leq c\kappa(\rho) \|\eta - \xi^T\|_{C^0([0, \frac{T}{2}], X)} \int_0^{T/2} e^{-\sigma\mu} d\sigma + c\kappa(\rho) \cdot 2\rho_0 \int_{T/2}^T e^{-\sigma\mu} d\sigma \\
& \quad + \rho_0 e^{-T\mu} + c\kappa(\rho) \|\eta\|_{\exp} \int_T^\infty e^{-\sigma\frac{3}{2}\mu} d\sigma. \\
& \leq \frac{c\kappa(\rho)}{\mu} \|\eta - \xi^T\|_{C^0([0, \frac{T}{2}], X)} + \frac{2c\kappa(\rho)}{\mu} \rho_0 e^{-T\frac{\mu}{2}} + \rho_0 e^{-T\mu} + \frac{2c\kappa(\rho)}{3\mu} \rho e^{-T\frac{3}{2}\mu} \\
& \leq \frac{1}{4} \|\eta - \xi^T\|_{C^0([0, \frac{T}{2}], X)} + 2\rho_0 e^{-T\frac{\mu}{2}}
\end{aligned}$$

where inequality two uses the exponential decay proposition 2.4 and the Lipschitz lemma 2.1 for f . We also used definition (21) of the \exp - T norm and the

⁴The argument relies on the right boundary of the t -interval running to $-\infty$, as $T \rightarrow \infty$. Therefore the right boundary of II needs to be strictly smaller than T , but at the same time be element of $[0, T]$ whichever T we pick. Thus any αT with $0 < \alpha < 1$ is a good choice.

⁵Exponential decay is achieved, if the left boundary of III is of the form αT with $\alpha > \frac{1}{2}$.

fact that the elements of Z^T take values in \mathcal{B}_{ρ_0} by step 1 and those of Z in $\mathcal{B}_\rho \subset \mathcal{B}_{\rho_0}$ by definition. Inequalities three and four are by calculation and (19).

(b) Pick $s \in [0, \frac{T}{2}]$. Similarly as in (a) we get that

$$\begin{aligned}
& \|\eta(s) - \xi^T(s)\|_X \\
& \leq \int_0^s \left\| e^{-(s-\sigma)A} \pi_+ \right\|_{\mathcal{L}(Y,X)} \|f \circ \eta(\sigma) - f \circ \xi^T(\sigma)\|_Y d\sigma \\
& \quad + \left(\int_s^{\frac{T}{2}} + \int_{\frac{T}{2}}^{\frac{3T}{4}} + \int_{\frac{3T}{4}}^T \right) \left\| e^{-(s-\sigma)A^-} \pi_- \right\|_{\mathcal{L}(Y,X)} \|f \circ \eta(\sigma) - f \circ \xi^T(\sigma)\|_Y d\sigma \\
& \quad + c e^{(s-T)\mu} \|\gamma\|_X + \int_T^\infty \left\| e^{-(s-\sigma)A^-} \pi_- \right\|_{\mathcal{L}(Y,X)} \|f \circ \eta(\sigma)\|_Y d\sigma \\
& \leq c \rho_0 e^{-T\frac{\mu}{2}} + c\kappa(\rho) \|\eta - \xi^T\|_{C^0([0, \frac{T}{2}], X)} \left(\int_0^s \frac{e^{-(s-\sigma)\mu}}{(s-\sigma)^{\frac{3}{4}}} d\sigma + \int_s^{\frac{T}{2}} e^{(s-\sigma)\mu} d\sigma \right) \\
& \quad + c\kappa(\rho) \int_{\frac{T}{2}}^{\frac{3T}{4}} e^{(s-\sigma)\mu} (\|\xi^T(\sigma) - \phi_\sigma(\gamma_T)\|_X + \|\phi_\sigma(\gamma_T)\|_X) d\sigma \\
& \quad + c\kappa(\rho) \int_{\frac{3T}{4}}^T e^{(s-\sigma)\mu} \|\xi^T(\sigma)\|_X d\sigma + c\kappa(\rho) \|\eta\|_{\exp} \int_{\frac{T}{2}}^\infty e^{(s-\frac{3}{2}\sigma)\mu} d\sigma.
\end{aligned}$$

Here we splitted the domain of integration in three parts in order to obtain overall exponential decay in T . In the last but one line presence of the term $\|\phi_\sigma(\gamma_T)\|$ requires the upper boundary of the domain of integration to be strictly less than T – otherwise $\tilde{\eta}(T) := \phi_T(\gamma_T) = \gamma$ is constant in T and not necessarily in \mathcal{B}_ρ as required by remark 2.8 to guarantee exponential decay. In the last line the domain of integration $\int_{T/2}^\infty$ is not a misprint.

To continue the estimate consider the last three lines. Now we explain how to get to the corresponding three lines in (39) below. Concerning line one recall that $s \in [0, T/2]$ and use (27) and (28). In line two use that $\|\xi^T(\sigma) - \phi_\sigma(\gamma_T)\|_X \leq \rho e^{-\sigma\frac{\mu}{2}}$ by definition of Z^T and that

$$\|\phi_\sigma(\gamma_T)\|_X = \|\tilde{\eta}(t)\|_X \leq e^{t\frac{\mu}{2}} \rho = \rho e^{(\sigma-T)\frac{\mu}{2}} \leq \rho e^{-T\frac{\mu}{8}} \quad (38)$$

by remark 2.8 for the backward time trajectory $\tilde{\eta}(t) = \phi_{\sigma-T}(\gamma)$ in the unstable manifold, where $t := \sigma - T$. Note that $t \in [-\frac{T}{2}, -\frac{T}{4}]$ due to $\sigma \in [\frac{T}{2}, \frac{3}{4}T]$. Hence $\tilde{\eta}(t) = \phi_t \gamma \in \phi_{-T/4} W_\varepsilon^u$ by invariance of the descending disk W_ε^u under the backward flow and the assumption that $\gamma \in S_\varepsilon^u = \partial W_\varepsilon^u$. But $\phi_{-T/4} W_\varepsilon^u \subset \mathcal{B}_\rho$ by definition of T_2 . Consequently $\tilde{\eta}$ takes values in \mathcal{B}_ρ and remark 2.8 indeed applies. To summarize, line two is bounded from above by

$$c\kappa(\rho) \cdot \rho \int_{\frac{T}{2}}^{\frac{3T}{4}} \left(e^{(s-\frac{3}{2}\sigma)\mu} + e^{(2s-\sigma-T)\frac{\mu}{2}} \right) d\sigma \leq c\kappa(\rho) \cdot \rho \left(\frac{2}{3\mu} e^{-T\frac{\mu}{4}} + \frac{2}{\mu} e^{-T\frac{\mu}{4}} \right).$$

In line three use that $\|\xi^T(\sigma)\|_X \leq \rho_0$ for any $\xi^T \in Z^T$ by step 1 and that

$\|\eta\|_{\text{exp}} \leq \rho$ by definition of Z . Then carry out the integrals to obtain

$$\begin{aligned} \|\eta(s) - \xi^T(s)\|_X &\leq c\rho_0 e^{-T\frac{\mu}{2}} + c\kappa(\rho) \left(\frac{8}{\mu^{\frac{1}{4}}} + \frac{1}{\mu} \right) \|\eta - \xi^T\|_{C^0([0, \frac{T}{2}], X)} \\ &\quad + \frac{8c\kappa(\rho)}{3\mu} \rho e^{-T\frac{\mu}{4}} + \frac{c\kappa(\rho)}{\mu} \rho_0 e^{-T\frac{\mu}{4}} + \frac{2c\kappa(\rho)}{3\mu} \rho e^{-T\frac{\mu}{4}} \quad (39) \\ &\leq \frac{1}{4} \|\eta - \xi^T\|_{C^0([0, \frac{T}{2}], X)} + 3c\rho_0 e^{-T\frac{\mu}{4}}. \end{aligned}$$

The last step uses smallness (19) of ρ . Take the sup over $s \in [0, \frac{T}{2}]$ to get

$$\|\eta - \xi^T\|_{C^0([0, \frac{T}{2}], X)} \leq 4c\rho_0 e^{-T\frac{\mu}{4}}. \quad (40)$$

Hence $\|\mathcal{G}^\infty(z_+) - \mathcal{G}_\gamma^T(z_+)\|_X = \|\eta(0) - \xi^T(0)\|_X \leq 3c\rho_0 e^{-T\frac{\mu}{4}}$, for all $\gamma \in S_\varepsilon^u$, times $T \geq T_2$, and $z_+ \in \mathcal{B}^+$ and this proves step 6. \square

This concludes the proof of the backward λ -lemma, theorem 1.2.

3.2 Proof of uniform C^1 convergence (theorem 1.3)

Since theorem 1.3 builds on the assumptions of the backward λ -lemma, theorem 1.2, we may use any of the six steps of its proof. The proof at hand is in two steps I and II. Fix $p = 1$, see remark 3.2, and pick $\gamma \in S_\varepsilon^u$ and $z_+ \in \mathcal{B}^+$.

Step I. (L^2 extension) $\|d\mathcal{G}_\gamma^T(z_+)v\|_2 \leq 2\|v\|_2$ for all $v \in Y^+ \cap L^2$ and $T \geq T_1$.

Proof. Pick $v \in X^+$ and $\tau \geq 0$ small. Consider the fixed point $\xi_{z_+ + \tau v} = \xi_{\gamma, z_+ + \tau v}^T \in Z^T$ of $\Psi_{\gamma, z_+ + \tau v}^T$. By (23) the fixed point property means that

$$\begin{aligned} \xi_{z_+ + \tau v}(s) &= e^{-sA} (z_+ + \tau v) + \int_0^s e^{-(s-\sigma)A} \pi_+ f(\xi_{z_+ + \tau v}(\sigma)) d\sigma \\ &\quad + e^{-(s-T)A^-} \gamma - \int_s^T e^{-(s-\sigma)A^-} \pi_- f(\xi_{z_+ + \tau v}(\sigma)) d\sigma \end{aligned} \quad (41)$$

for every $s \in [0, T]$. By the proof of step 3 the composition of maps $\tau \mapsto \xi_{z_+ + \tau v} \mapsto \xi_{z_+ + \tau v}(s)$ is of class C^1 . Hence the linearization

$$X_v(s) = X_{\gamma, z_+}^T(s)v := \frac{d}{d\tau} \Big|_{\tau=0} \xi_{z_+ + \tau v}(s)$$

is well-defined. It satisfies

$$\begin{aligned} X_v(s) &= e^{-sA} v + \int_0^s e^{-(s-\sigma)A} \pi_+ \left(df|_{\xi_{z_+}(\sigma)} \circ X_v(\sigma) \right) d\sigma \\ &\quad - \int_s^T e^{-(s-\sigma)A^-} \pi_- \left(df|_{\xi_{z_+}(\sigma)} \circ X_v(\sigma) \right) d\sigma \end{aligned} \quad (42)$$

for every $s \in [0, T]$. Use (30) to see that $X_v(0) = \frac{d}{d\tau} \Big|_{\tau=0} \xi_{z_+ + \tau v}(0) = d\mathcal{G}_\gamma^T(z_+)v$. To conclude the proof it remains to show that $\|X_v(0)\|_2 \leq 2\|v\|_2$. Recall that

$$\|e^{-sA} \pi_+\|_{\mathcal{L}(L^2, W^{1,2})} \leq cs^{-\frac{1}{2}} e^{-s\mu}, \quad s > 0, \quad (43)$$

by proposition 2.4. This motivates to define the weighted exp norm

$$\|X_v\|_{\frac{1}{2}, \text{exp}} = \|X_v\|_{\frac{1}{2}, \text{exp}, T} := \sup_{s \in [0, T]} s^{\frac{1}{2}} e^{s\frac{\mu}{2}} \|X_v(s)\|_X.$$

As we show next this choice allows to estimate $\|X_v(s)\|_X$ up to a singular factor in terms of $\|v\|_2$ instead of $\|v\|_X$. By (42) and $v \in X^+ \subset W^{1,2} \hookrightarrow L^2$ we obtain

$$\begin{aligned} s^{\frac{1}{2}} e^{s\frac{\mu}{2}} \|X_v(s)\|_X &\leq s^{\frac{1}{2}} e^{s\frac{\mu}{2}} \|e^{-sA} \pi_+ \|_{\mathcal{L}(L^2, W^{1,2})} \|v\|_2 \\ &\quad + s^{\frac{1}{2}} e^{s\frac{\mu}{2}} \int_0^s \|e^{-(s-\sigma)A} \pi_+ \|_{\mathcal{L}(Y, X)} \kappa(\rho) \|X_v(\sigma)\|_X d\sigma \\ &\quad + s^{\frac{1}{2}} e^{s\frac{\mu}{2}} \int_s^T \|e^{-(s-\sigma)A^-} \pi_- \|_{\mathcal{L}(Y, X)} \kappa(\rho) \|X_v(\sigma)\|_X d\sigma \\ &\leq c e^{-s\frac{\mu}{2}} \|v\|_2 + c \kappa(\rho) \|X_v\|_{\frac{1}{2}, \text{exp}} \int_0^s \frac{e^{-(s-\sigma)\frac{\mu}{2}}}{(s-\sigma)^{\frac{3}{4}}} \left(\frac{s}{\sigma}\right)^{\frac{1}{2}} d\sigma \\ &\quad + c \kappa(\rho) \|X_v\|_{\frac{1}{2}, \text{exp}} \int_s^T e^{\frac{3}{2}(s-\sigma)\mu} \left(\frac{s}{\sigma}\right)^{\frac{1}{2}} d\sigma \\ &\leq c e^{-s\frac{\mu}{2}} \|v\|_2 + c \kappa(\rho) \left(\frac{18}{\mu^{\frac{1}{4}}} + \frac{2}{3\mu}\right) \|X_v\|_{\frac{1}{2}, \text{exp}} \end{aligned}$$

for every $s \in [0, T]$. Concerning inequality one we used that $\xi_{z_+} \in Z^T$ takes values in $B_{\rho_0} \subset \mathcal{U}$ by step 1. Hence corollary 2.2 applies and provides the estimate for df . In inequality two we used that $\|X_v(\sigma)\|_X \leq \sigma^{-\frac{1}{2}} e^{-\sigma\frac{\mu}{2}} \|X_v\|_{\frac{1}{2}, \text{exp}}$ by definition of the exp norm. We used (43) to obtain the first term and proposition 2.4 to obtain the other two terms of the sum. Inequality three will be proved below. Now use smallness (19) of ρ and take the supremum over $s \in [0, T]$ to obtain

$$\|X_v\|_{\frac{1}{2}, \text{exp}} \leq 2c \|v\|_2. \quad (44)$$

Concerning inequality three we need to estimate the two integrals. Observe that $\int_s^T e^{\frac{3}{2}(s-\sigma)\mu} \left(\frac{s}{\sigma}\right)^{\frac{1}{2}} d\sigma \leq \int_s^T e^{\frac{3}{2}(s-\sigma)\mu} d\sigma \leq \frac{2}{3\mu}$ and

$$\begin{aligned} &\int_0^s \frac{e^{-(s-\sigma)\frac{\mu}{2}}}{(s-\sigma)^{\frac{3}{4}}} \left(\frac{s}{\sigma}\right)^{\frac{1}{2}} d\sigma \\ &= \int_0^{s/2} \underbrace{e^{-(s-\sigma)\frac{\mu}{2}}}_{\leq e^{-s\frac{\mu}{4}}} \underbrace{(s-\sigma)^{-\frac{3}{4}} s^{\frac{1}{2}} \sigma^{-\frac{1}{2}}}_{\geq s/2} d\sigma + \int_{s/2}^s \frac{e^{-(s-\sigma)\frac{\mu}{2}}}{(s-\sigma)^{\frac{3}{4}}} \underbrace{\left(\frac{s}{\sigma}\right)^{\frac{1}{2}}}_{\leq 2} d\sigma \quad (45) \\ &\leq 2^{\frac{3}{4}} s^{-\frac{1}{4}} e^{-s\frac{\mu}{4}} \int_0^{s/2} \sigma^{-\frac{1}{2}} d\sigma + \frac{4}{\mu^{\frac{1}{4}}} \Gamma\left(\frac{1}{4}\right) \leq \frac{2}{\mu^{\frac{1}{4}}} + \frac{16}{\mu^{\frac{1}{4}}}. \end{aligned}$$

Here we used that the last integral is equal to $\sqrt{2s}$ and $h(s) := 2^{\frac{5}{4}} s^{\frac{1}{4}} e^{-s\frac{\mu}{4}}$ is bounded by $h(s_{\max}) = h(1/\mu) = 2(2/\mu e)^{1/4}$. Furthermore, we used (28).

We start over again estimating $X_v(s)$, but now at $s = 0$ and in the L^2 norm. Similarly as above, using that $\|X_v(\sigma)\|_X \leq 2c\sigma^{-\frac{1}{2}}e^{-\sigma\frac{\mu}{2}}\|v\|_2$ by (44) we get

$$\begin{aligned}\|X_v(0)\|_2 &\leq \|v\|_2 + \int_0^T \left\| e^{\sigma A^-} \pi_- \right\|_{\mathcal{L}(L^1, L^2)} \kappa(\rho) \|X_v(\sigma)\|_X d\sigma \\ &\leq \|v\|_2 + 2c^2 \kappa(\rho) \|v\|_2 \int_0^T e^{-\frac{3}{2}\sigma\mu} \sigma^{-\frac{1}{2}} d\sigma \\ &\leq \|v\|_2 + 2c^2 \kappa(\rho) \left(4 + \frac{8}{9\mu} \right) \|v\|_2 \leq 2\|v\|_2\end{aligned}$$

whenever $s \in [0, T]$. In inequality one we also used that e^{-sA} is a strongly continuous semigroup on L^2 by proposition 2.4 and that $\|\cdot\|_{\mathcal{L}(L^1, L^2)} \leq \|\cdot\|_{\mathcal{L}(L^1, W^{1,2})}$ by the embedding $W^{1,2} \hookrightarrow L^2$. The final inequality is by smallness (19) of ρ . Inequality three uses the following consequence (case $s = 0$ and with μ replaced by $3\mu/2$) of Hölder's inequality (domain $[s, \infty)$, $p = 3$, $q = 3/2$), namely

$$\int_s^\infty e^{-\sigma\mu} \sigma^{-\frac{1}{2}} d\sigma \leq \|e^{-\sigma\frac{\mu}{2}}\|_{L^3} \|e^{-\sigma\frac{\mu}{2}} \sigma^{-\frac{1}{2}}\|_{L^{3/2}} \leq \left(4 + \frac{4}{3\mu} \right) e^{-s\frac{\mu}{2}} \quad (46)$$

for $s \geq 0$. Step two uses that $\|e^{-\sigma\frac{\mu}{2}}\|_{L^3} = (2/3\mu)^{1/3} e^{-s\frac{\mu}{2}}$ by calculation and that $\|e^{-\sigma\frac{\mu}{2}} \sigma^{-\frac{1}{2}}\|_{L^{3/2}}^{3/2} = \int_s^\infty e^{-\sigma\frac{3}{4}\mu} \sigma^{-\frac{3}{4}} d\sigma \leq \int_0^1 \sigma^{-\frac{3}{4}} d\sigma + \int_1^\infty e^{-\sigma\frac{3}{4}\mu} d\sigma = 4 + \frac{4}{3\mu} e^{-\frac{3}{4}\mu}$. This concludes the proof of Step I. \square

Step II. $\|d\mathcal{G}_\gamma^T(z_+)v - d\mathcal{G}^\infty(z_+)v\|_2 \leq 3ce^{-T\frac{\mu}{4}}\|v\|_2 \quad \forall T \geq T_0 \quad \forall v \in \pi_+(L^2)$.

Proof. The proof of convergence of the linearized graph maps should use convergence of the graph maps themselves. Indeed (40) is a key ingredient. Another one is the Lipschitz estimate for df provided by lemma 2.1.

Pick $T \geq T_0$ and $v \in X^+$. Consider the fixed point $\xi_{z_+} = \xi_{\gamma, z_+}^T$ of the strict contraction Ψ_{γ, z_+}^T on Z^T and the fixed point η_{z_+} of Ψ_{z_+} on Z . It is a side remark that theorem 2.6 is recovered by the present setup for $T = \infty$ and $\gamma := 0$. For $\tau \geq 0$ small $\xi_{z_+ + \tau v}$ satisfies the integral equation (41) and $\eta_{z_+ + \tau v}$ satisfies (41) with $T = \infty$; in particular, term three in that sum disappears. Consider the linearizations $X_v := \frac{d}{d\tau}|_{\tau=0} \xi_{\gamma, z_+ + \tau v}^T$ and $Y_v := \frac{d}{d\tau}|_{\tau=0} \eta_{z_+ + \tau v}$ and observe that X_v satisfies the integral equation (42) and Y_v satisfies (42) with $T = \infty$. We know that $d\mathcal{G}_\gamma^T(z_+)v = X_v(0)$ by the identity following (42), similarly $d\mathcal{G}^\infty(z_+)v = Y_v(0)$. It remains to estimate $\|X_v(0) - Y_v(0)\|_2$. Define

$$\|X_v\|_* := \sup_{s \in [0, \frac{1}{2}T]} s^{\frac{1}{2}} \|X_v(s)\|_X$$

and abbreviate $\xi := \xi_{\gamma, z_+}^T$ and $\eta := \eta_{z_+}$. Then we obtain the L^2 estimate

$$\begin{aligned}\|X_v(0) - Y_v(0)\|_2 &\leq \left(\int_0^{\frac{1}{2}T} + \int_{\frac{1}{2}T}^T \right) \left\| e^{\sigma A^-} \pi_- \right\|_{\mathcal{L}(L^1, L^2)} \|df|_{\xi(\sigma)} \circ X_v(\sigma) - df|_{\eta(\sigma)} \circ Y_v(\sigma)\|_Y d\sigma \\ &\quad + \int_T^\infty \left\| e^{\sigma A^-} \pi_- \right\|_{\mathcal{L}(L^1, L^2)} \|df|_{\eta(\sigma)} \circ Y_v(\sigma)\|_Y d\sigma\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^{\frac{1}{2}T} ce^{-\sigma\mu} \left(\underbrace{\kappa_* \|\xi(\sigma) - \eta(\sigma)\|_X}_{\leq 4c\rho_0 e^{-T\frac{\mu}{4}}, (40)} \|X_v(\sigma)\|_X + \kappa(\rho) \|X_v(\sigma) - Y_v(\sigma)\|_X \right) d\sigma \\
&\quad + \int_{\frac{1}{2}T}^T ce^{-\sigma\mu} \kappa(\rho) \underbrace{\|X_v(\sigma)\|_X}_{\leq 2c\sigma^{-\frac{1}{2}} e^{-\sigma\frac{\mu}{2}} \|v\|_2} d\sigma + \int_{\frac{1}{2}T}^{\infty} ce^{-\sigma\mu} \kappa(\rho) \underbrace{\|Y_v(\sigma)\|_X}_{\leq 2c\sigma^{-\frac{1}{2}} e^{-\sigma\frac{\mu}{2}} \|v\|_2} d\sigma \\
&\leq 8\rho_0 c^3 \kappa_* e^{-T\frac{\mu}{2}} \|v\|_2 \int_0^{\frac{1}{2}T} e^{-\sigma\frac{3}{2}\mu} \sigma^{-\frac{1}{2}} d\sigma + c\kappa(\rho) \|X_v - Y_v\|_* \int_0^{\frac{1}{2}T} e^{-\sigma\mu} \sigma^{-\frac{1}{2}} d\sigma \\
&\quad + 4c^2 \kappa(\rho) \|v\|_2 \int_{\frac{1}{2}T}^{\infty} e^{-\sigma\frac{3}{2}\mu} \sigma^{-\frac{1}{2}} d\sigma.
\end{aligned}$$

Inequality two uses that by the Lipschitz lemma 2.1 for df and its corollary 2.2

$$\begin{aligned}
&\|df|_{\xi(\sigma)} \circ X_v(\sigma) - df|_{\eta(\sigma)} \circ Y_v(\sigma)\|_Y \\
&= \|(df|_{\xi(\sigma)} - df|_{\eta(\sigma)}) \circ X_v(\sigma) + df|_{\eta(\sigma)} \circ (X_v(\sigma) - Y_v(\sigma))\|_Y \\
&\leq \kappa_* \|\xi(\sigma) - \eta(\sigma)\|_X \|X_v(\sigma)\|_X + \kappa(\rho) \|X_v(\sigma) - Y_v(\sigma)\|_X
\end{aligned}$$

and $\|df|_{\eta(\sigma)} \circ Y_v(\sigma)\|_Y \leq \kappa(\rho) \|Y_v(\sigma)\|_X$, respectively. The integral over $[\frac{1}{2}T, T]$ was treated by the triangle inequality and its η part has been incorporated into the integral over $[\frac{1}{2}T, \infty)$. Note that no factor $\sigma^{-\frac{1}{4}}$ is caught by the integrals after the second inequality sign, because π_- projects onto the finite dimensional space Y^- . More precisely, use that $\|\cdot\|_{\mathcal{L}(L^1, L^2)} \leq \|\cdot\|_{\mathcal{L}(L^1, W^{1,2})}$ by the embedding $W^{1,2} \hookrightarrow L^2$, then apply proposition 2.4. Consider inequality three. In the calculation above we indicated how to estimate certain terms. The estimates used are (40) and (44). We also used that (44) holds for Y_v with $T = \infty$.

Now use (46) for the integrals and (17) for ρ_0 and (19) for ρ to get that

$$\|X_v(0) - Y_v(0)\|_2 \leq \frac{1}{4} \|X_v - Y_v\|_* + 2ce^{-\frac{3}{8}T\mu} \|v\|_2. \quad (47)$$

It remains to prove exponential decay of the weighted sup norm $\|\cdot\|_*$ over the domain $[0, \frac{1}{2}T]$. Fix $s \in [0, \frac{1}{2}T]$ and conclude similarly as above that

$$\begin{aligned}
&s^{\frac{1}{2}} \|X_v(s) - Y_v(s)\|_X \\
&\leq s^{\frac{1}{2}} c \int_0^s \frac{e^{-(s-\sigma)\mu}}{(s-\sigma)^{\frac{3}{4}}} \left(\underbrace{\kappa_* \|\xi(\sigma) - \eta(\sigma)\|_X}_{\leq 4c\rho_0 e^{-T\frac{\mu}{4}}} \underbrace{\|X_v(\sigma)\|_X}_{\leq 2c\sigma^{-\frac{1}{2}} e^{-\sigma\frac{\mu}{2}} \|v\|_2} \right. \\
&\quad \left. + \kappa(\rho) \|X_v(\sigma) - Y_v(\sigma)\|_X \right) d\sigma \\
&\quad + s^{\frac{1}{2}} c \int_s^{\frac{T}{2}} e^{(s-\sigma)\mu} \left(\kappa_* \|\xi(\sigma) - \eta(\sigma)\|_X \|X_v(\sigma)\|_X \right. \\
&\quad \left. + \kappa(\rho) \|X_v(\sigma) - Y_v(\sigma)\|_X \right) d\sigma \\
&\quad + s^{\frac{1}{2}} c \kappa(\rho) \int_{\frac{T}{2}}^T e^{(s-\sigma)\mu} \|X_v(\sigma)\|_X d\sigma + s^{\frac{1}{2}} c \kappa(\rho) \int_{\frac{T}{2}}^{\infty} e^{(s-\sigma)\mu} \|Y_v(\sigma)\|_X d\sigma
\end{aligned}$$

$$\begin{aligned}
&\leq 8\rho_0 c^3 \kappa_* e^{-T\frac{\mu}{4}} \|v\|_2 \left(\int_0^s \frac{e^{-(s-\sigma)\mu} e^{-\sigma\frac{\mu}{2}} s^{\frac{1}{2}}}{(s-\sigma)^{\frac{3}{4}} \sigma^{\frac{1}{2}}} d\sigma + \int_s^{\frac{T}{2}} \frac{e^{(s-\sigma)\mu} e^{-\sigma\frac{\mu}{2}} s^{\frac{1}{2}}}{\sigma^{\frac{1}{2}}} d\sigma \right) \\
&\quad + c\kappa(\rho) \|X_v - Y_v\|_* \left(\int_0^s \frac{e^{-(s-\sigma)\mu} s^{\frac{1}{2}}}{(s-\sigma)^{\frac{3}{4}} \sigma^{\frac{1}{2}}} d\sigma + \int_s^{\frac{T}{2}} \frac{e^{(s-\sigma)\mu} s^{\frac{1}{2}}}{\sigma^{\frac{1}{2}}} d\sigma \right) \\
&\quad + 4c^2 \kappa(\rho) \|v\|_2 \int_{\frac{T}{2}}^\infty e^{(s-\frac{3}{2}\sigma)\mu} \left(\frac{s}{\sigma}\right)^{\frac{1}{2}} d\sigma \\
&\leq \left(8c^2 \rho_0 \kappa_* \left(\frac{18}{\mu^{\frac{1}{4}}} + \frac{1}{\mu} \right) + 4c\kappa(\rho) \frac{2}{3\mu} \right) c \|v\|_2 e^{-T\frac{\mu}{4}} \\
&\quad + c\kappa(\rho) \left(\frac{18}{\mu^{\frac{1}{4}}} + \frac{1}{\mu} \right) \|X_v - Y_v\|_*.
\end{aligned}$$

It is a side remark that without the weight factor $s^{1/2}$ in the $\|\cdot\|_*$ norm the integrals involving $(s-\sigma)^{-3/4}$ cause trouble concerning boundedness for s close to zero. It is another side remark that due to the presence of the extra factor $\|X_v(\sigma)\|_X$ we do not have to cut the interval $[\frac{T}{2}, T]$ into two pieces as we did in step 6 in the proof of theorem 1.2. In inequality three we used the following estimates. By (45) and by calculation, respectively, we obtain

$$\int_0^s \frac{e^{-(s-\sigma)\mu}}{(s-\sigma)^{\frac{3}{4}}} \left(\frac{s}{\sigma}\right)^{\frac{1}{2}} d\sigma \leq \frac{18}{\mu^{\frac{1}{4}}}$$

and

$$\int_s^{\frac{T}{2}} e^{(s-\sigma)\mu} \left(\frac{s}{\sigma}\right)^{\frac{1}{2}} d\sigma \leq \frac{1}{\mu}.$$

To get the second of these estimates we used $s/\sigma \leq 1$. Again by calculation

$$\int_{\frac{T}{2}}^\infty e^{(s-\frac{3}{2}\sigma)\mu} \left(\frac{s}{\sigma}\right)^{\frac{1}{2}} d\sigma \leq \int_{\frac{T}{2}}^\infty e^{(s-\frac{3}{2}\sigma)\mu} d\sigma \leq \frac{2}{3\mu} e^{(s-\frac{3}{2}T)\mu} \leq \frac{2}{3\mu} e^{-T\frac{\mu}{4}},$$

since $s \leq T/2 \leq \sigma$. Now use smallness (17) of ρ_0 and (19) of ρ to get

$$s^{\frac{1}{2}} \|X_v(s) - Y_v(s)\|_X \leq \frac{1}{4} \|X_v - Y_v\|_* + 3ce^{-T\frac{\mu}{4}} \|v\|_2.$$

Take the supremum over $s \in [0, \frac{1}{2}T]$ to obtain

$$\|X_v - Y_v\|_* \leq 4ce^{-T\frac{\mu}{4}} \|v\|_2.$$

By estimate (47) this concludes the proof of Step II. \square

The proof of theorem 1.3 is complete.

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